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2008 J. Phys. A: Math. Theor. 41 485303

(http://iopscience.iop.org/1751-8121/41/48/485303)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 485303 (35pp) doi:10.1088/1751-8113/41/48/485303

Finite-gap systems, tri-supersymmetry and self-isospectrality

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Received 30 June 2008, in final form 27 September 2008
Published 22 October 2008
Online at stacks.iop.org/JPhysA/41/485303

Abstract

We show that an n-gap periodic quantum system with parity-even smooth potential admits $2^n - 1$ isospectral super-extensions. Each is described by a tri-supersymmetry that originates from a higher-order differential operator of the Lax pair and two-term nonsingular decompositions of it; its local part corresponds to a spontaneously partially broken centrally extended nonlinear $N = 4$ supersymmetry. We conjecture that any finite-gap system having antiperiodic singlet states admits a self-isospectral tri-supersymmetric extension with the partner potential to be the original one translated for a half-period. Applying the theory to a broad class of finite-gap elliptic systems described by a two-parametric associated Lamé equation, our conjecture is supported by the explicit construction of the self-isospectral tri-supersymmetric pairs. We find that the spontaneously broken tri-supersymmetry of the self-isospectral periodic system is recovered in the infinite-period limit.

PACS numbers: 11.30.Pb, 02.30.Ik, 11.30.Na, 03.65.−w

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Finite-gap periodic quantum systems play an important role in physics. They underlie the theory of periodic solutions in nonlinear integrable systems, including the Korteweg–de Vries, the nonlinear Schrödinger, the Kadomtsev–Petviashvili and the sine-Gordon equations [1–5]. Being analytically solvable systems, they find various applications in diverse areas. The list of their applications is extensive, and among others includes the modeling of crystals [6–8], the theory of monopoles [9], instantons and sphalerons [10, 11], classical Ginzburg–Landau field theory [12], Josephson junctions theory [13], magnetostatic problems [14], inhomogeneous

cosmologies [15], Kaluza–Klein theories [16], chaos [17], preheating after inflation modern theories [18], string theory [19], matrix models [20], supersymmetric Yang–Mills theory [21, 22] and AdS/CFT duality [23].

Some time ago it was showed by Braden and Macfarlane [24], and in a more broad context by Dunne and Feinberg [25], that a usual $N = 2$ supersymmetric extension of a periodic quantum system may produce a completely isospectral pair with a *zero-energy doublet* of the ground states. Such a picture is completely different from that taking place in non-periodic systems described by the same linear $N = 2$ superalgebraic structure $\{Q_a, Q_b\} = 2\delta_{ab}H, [Q_a, H] = 0$. There, the complete isospectrality of the superpartners happens only in the case of a spontaneously broken supersymmetry, characterized by a positive energy of the lowest supersymmetric doublet [26]. Furthermore, it was showed that there exist peculiar isospectral supersymmetric periodic systems, in which the partner potentials are identical in shape but mutually translated for a half-period, or reflected, or translated and reflected. A pair of superpartner potentials with such a property was named by Dunne and Feinberg as *self-isospectral*. The phenomenon of self-isospectrality with a half-period shift was illustrated by some examples of exactly soluble models belonging to a class of finite-gap periodic systems. Later on isospectral and self-isospectral supersymmetric finite-gap periodic systems were studied in various aspects [27–29], and it was found in [30] that a property of the self-isospectrality may also appear in some periodic finite-gap systems based on a nonlinear supersymmetry of the second order $\{Q_a, Q_b\} = 2\delta_{ab}P_2(H), [Q_a, H] = 0$ [31–33], where $P_2(H)$ is a quadratic polynomial. The nature and origin of isospectrality and self-isospectrality in finite-gap systems have remained, however, to be obscure.

Recently, we showed [34] that self-isospectrality may be realized by a non-relativistic electron in the periodic magnetic and electric fields of a special form, and indicated on a peculiar nonlinear supersymmetric structure associated with it.

In the present paper, we study superextension of quantum periodic systems with a parity-even smooth finite-gap potential of general form, and show that it is characterized by an unusual tri-supersymmetric structure. This peculiar supersymmetric structure originates from the higher-order differential operator of the Lax pair, and its decomposability in pairs of nonsingular operators. The superalgebra, generated by three indicated integrals of motion together with trivial integrals associated with parity symmetry and matrix extension, has a nonlinear nature, that reflects a nonlinearity of a spectral polynomial of the original finite-gap system. The higher-order operator of the Lax pair of a nontrivial n-gap ($n > 0$) system admits $2^n - 1$ nonsingular two-term decompositions. By means of the Crum–Darboux transformation, with each nonsingular decomposition we associate a particular tri-supersymmetric extension, and as a result get a family of 2^n completely isospectral systems. We show that a local part of the tri-supersymmetry is a spontaneously partially broken centrally extended nonlinear $N = 4$ supersymmetry, which explains the nature and origin of the complete isospectrality. When the original finite-gap system has in its spectrum a nonzero number of anti-periodic singlet states corresponding to the edges of permitted bands, among all the non-singular decompositions of the higher-order operator of the Lax pair there is a special one which corresponds to a separation of all the singlets into orthogonal subspaces of periodic and anti-periodic states. We conjecture that it is this separation that produces a self-isospectral tri-supersymmetric system. This means particularly that all the set of 2^n completely isospectral systems we get, including the original n-gap system with the specified special property, is divided into 2^{n-1} self-isospectral tri-supersymmetric pairs. Then we apply a general theory to a broad class of finite-gap elliptic (double periodic) quantum systems described by a two-parametric family of associated Lamé equations. Any such system has in its spectrum a non-empty subspace of anti-periodic singlet states, and we support our conjecture by the explicit construction of the

self-isospectral tri-supersymmetric pairs. We also investigate a rather intricate picture of the infinite-period limit of the tri-supersymmetry.

The paper is organized as follows. In the following section, we first discuss general properties of the finite-gap periodic systems with a smooth potential, and show that any parity-even n-gap system is characterized by a hidden bosonized $N = 2$ nonlinear supersymmetry of order $2n + 1$. This supersymmetry reflects the peculiarities of the band structure. In section 3, we show how the tri-supersymmetric extensions of the system are constructed by means of the Crum–Darboux transformation. There we also investigate a general structure and properties of the tri-supersymmetry. In section 4, we apply a general theory to the case of finite-gap elliptic systems described by the associated Lamé equation. In section 5, the infinite-period limit of the tri-supersymmetry is studied. Section 6 is devoted to concluding remarks and outlook.

2. Hidden supersymmetry in finite-gap systems

To have a self-contained presentation, in this section we first summarize briefly the properties of the quantum periodic systems of a general form. Then we restrict the consideration to the case of the smooth parity-even finite-gap systems to reveal in them a hidden bosonized nonlinear supersymmetry whose order is defined by the number of energy gaps.

2.1. General properties of quantum periodic systems

Consider a quantum system given by a Hamiltonian operator $H = -D^2 + u(x)$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$, with a real *smooth* periodic potential $u(x)$, $u(x) = u(x + 2L)$. For the corresponding stationary Schrödinger equation,

$$H\Psi(x) = E\Psi(x), \tag{2.1}$$

known in the literature as Hill's equation, we choose some real basis of solutions, $\psi_1(x; E)$, $\psi_2(x; E)$. The operator of translation for the period $2L$, or the monodromy operator,

$$T\Psi(x) = \Psi(x + 2L), \tag{2.2}$$

commutes with the Hamiltonian H, $[T, H] = 0$. It preserves a two-dimensional linear vector space of solutions of (2.1), and can be represented there by the second-order monodromy matrix $M(E)$,

$$T\psi_a(x; E) = \psi_a(x + 2L; E) = M_{ab}(E)\psi_b(x; E). \tag{2.3}$$

The change of the basis, $\psi_a(x; E) \to \tilde{\psi}_a(x; E) = A_{ab}\psi_b(x; E)$, $\det A \neq 0$, generates a conjugation of the monodromy matrix, $M(E) \to \tilde{M}(E) = AM(E)A^{-1}$, but do not change its determinant, $\det M(E) = \det \tilde{M}(E)$, trace, $\operatorname{Tr} M(E) = \operatorname{Tr} \tilde{M} \equiv \mathcal{D}(E)$, and eigenvalues, given by solutions of the characteristic equation

$$\det(M(E) - \mu I) = 0. \tag{2.4}$$

Let us choose a particular basis of solutions fixed by conditions

$$\psi_1(0; E) = 1, \qquad \psi_1'(0; E) = 0, \qquad \psi_2(0; E) = 0, \qquad \psi_2'(0; E) = 1, \tag{2.5}$$

where prime denotes the x-derivative. Differentiating relation (2.3) in x and putting then $x = 0$ in (2.3) and in the derived relation, we find the form of the monodromy matrix in basis (2.5),

$$M(E) = \begin{pmatrix} \psi_1(2L; E) & \psi_1'(2L; E) \\ \psi_2(2L; E) & \psi_2'(2L; E) \end{pmatrix}. \tag{2.6}$$

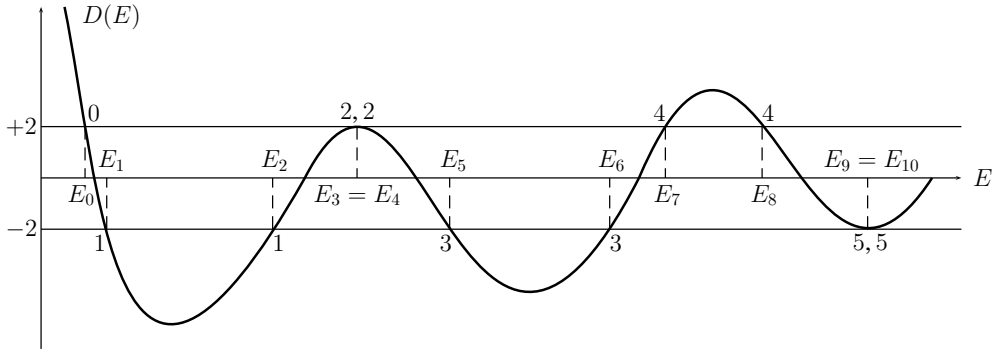


Figure 1. The discriminant $\mathcal{D}(E)$ in a generic situation of a periodic potential.

Wronskian $W(\psi_1, \psi_2) = \psi_1\psi_2' - \psi_1'\psi_2$ of any two linearly independent solutions of equation (2.1) takes a nonzero x-independent value, which for basis (2.5) is equal to 1. Then the explicit form of the real monodromy matrix (2.6) shows that a basis-independent value of its determinant does not depend on energy either, $\det M(E) = 1$, and so, $M(E) \in sl(2, \mathbb{R})$. Note that the change $x = 0 \to x_0 \in \mathbb{R}$ in relations (2.5) gives a one-parametric family of the bases, $\psi_a(x; x_0, E) = A(x_0)_{ab}\psi_b(x; E)$, $A(x_0) \in sl(2, \mathbb{R})$, playing an important role in the theory of periodic quantum systems [1–4]. In such a basis, the monodromy matrix will include an additional dependence on the marked point x_0, $M(E, x_0) \in sl(2, \mathbb{R})$.

With taking into account that $\det M = 1$, the characteristic equation (2.4) is reduced to $1 - \mathcal{D}(E)\mu + \mu^2 = 0$, and the basis-independent eigenvalues of the monodromy matrix are given in terms of its trace[1],

$$\mu_{1,2}(E) = \tfrac{1}{2}\mathcal{D}(E) \pm \sqrt{\mathcal{D}(E)^2/4 - 1}. \tag{2.7}$$

In correspondence with $\det M(E) = 1$, $\mu_1\mu_2 = 1$. Common eigenstates of H and T are described by the Bloch–Floquet functions $\psi_\pm(x; E)$, which satisfy a relation

$$T\psi_\pm(x; E) = \exp(\pm \mathrm{i}\kappa(E))\psi_\pm(x; E), \tag{2.8}$$

where $\mu_{1,2}(E) = \exp(\pm \mathrm{i}\kappa(E))$, and the quasi-momentum $\kappa(E)$ is given by

$$2\cos\kappa(E) = \mathcal{D}(E). \tag{2.9}$$

The values of the discriminant $\mathcal{D}(E)$ define the spectral properties of the periodic Schrödinger equation. For some energies $E \in (E_{2i-1}, E_{2i})$, $i = 0, \ldots, E_i < E_{i+1}$, $E_{-1} = -\infty$, the quasi-momentum $\kappa(E)$ takes complex values, and $|\mathcal{D}(E)| > 2$. Solutions corresponding to such E's are not physically acceptable as they diverge in $x = -\infty$ or $+\infty$. For these values of E we have a forbidden band, or an energy gap, see figure 1. In a generic case, a periodic quantum system has an infinite number of gaps. The width of the gaps decreases rapidly when energy increases, while the rate of decrease depends on the smoothness of the potential. In the case of analytic potentials, the gaps decrease exponentially. Energies E for which $|\mathcal{D}(E)| \leqslant 2$, define permitted bands, or permitted zones. Here, the quasi-momentum $\kappa(E)$ takes real values, and complex numbers $\exp(\pm \mathrm{i}\kappa(E))$ have modulus equals to 1. All the energy levels with $|\mathcal{D}(E)| < 2$ are doubly degenerate, but for $|\mathcal{D}(E)| = 2$ we have two essentially different cases. For those E, which separate permitted and prohibited bands, corresponding eigenvalue of the monodromy matrix is *non-degenerate*, the matrix M has a

[1] The trace of the monodromy matrix is called in the literature the Lyapunov function, Hill determinant, or discriminant of the Schrödinger equation.

form of Jordan matrix, and a physical singlet band-edge state is periodic, $\exp(\mathrm{i}\kappa(E)) = +1$, if $\mathcal{D}(E) = 2$, while for $\mathcal{D}(E) = -2$ a singlet state is antiperiodic, $\exp(\mathrm{i}\kappa(E)) = -1$. When $|\mathcal{D}(E)| = 2$ but the corresponding eigenvalue of the monodromy matrix is doubly degenerate, M is diagonalizable on the two linearly independent Bloch–Floquet states, which both are periodic if $\mathcal{D}(E) = 2$, or are antiperiodic when $\mathcal{D}(E) = -2$. This second situation, that corresponds to points $E_3 = E_4$ and $E_9 = E_{10}$ on figure 1, takes place when a prohibited band disappears.

Summarizing, the interval $(-\infty, E_0)$ constitutes the lowest forbidden band. The permitted bands with $|\mathcal{D}(E)| \leqslant 2$ are separated by prohibited bands, or energy gaps. All the energy levels in the interior of permitted bands have a double degeneration, while the states at their edges are singlets.

According to the oscillation theorem [35], the common eingenstates of H and monodromy operator T with energies $E_0 < E_1 \leqslant E_2 < E_3 \leqslant E_4 < E_5 \leqslant E_6 < \cdots$ such that $|\mathcal{D}(E_k)| = 2$ are described by the wavefunctions which are characterized by the periods $2L, 4L, 4L, 2L, 2L, 4L, 4L, \ldots$, and by the node numbers in the period $2L$ equal to $0, 1, 1, 2, 2, 3, 3, \ldots$, see figure 1. The odd number of nodes corresponds to antiperiodic states, whereas the periodic states have an even number of nodes in the period $2L$. The singlet states at the edges of the same prohibited band have the same number of nodes and the same periodicity, and their nodes are alternating.

2.2. *Finite-gap systems and hidden bosonized supersymmetry*

In some periodic potentials infinite number of bands merge together so that only finite number of gaps remains in the spectrum. Such potentials are called finite gap. The simplest case of a zero-gap system corresponds here to a free particle with $u(x) = \mathrm{const}$[2]. For the Schrödinger equation with a finite-gap potential the spectrum and eigenfunctions can be presented in an analytical form[3]. Having also in mind that for analytical potentials the size of the gaps decreases exponentially when energy increases, any periodic potential can be approximated by a finite-gap potential if narrow gaps are disregarded.

From now on we suppose that a periodic potential $u(x)$ is finite gap. *Additionally*, we assume that it is an *even* function, $u(x) = u(-x)$. Then a reflection (parity) operator R, $R\psi(x) = \psi(-x)$, is a nonlocal integral of motion, $[R, H] = 0$. Periodicity and parity symmetry together imply that the potential possesses also a middle-point reflection symmetry $u(L + x) = u(L - x)$.

The spectrum $\sigma(H)$ of a nontrivial n-gap ($n > 0$) system is characterized by the band structure, $\sigma(H) = [E_0, E_1] \cup \cdots \cup [E_{2n-2}, E_{2n-1}] \cup [E_{2n}, \infty)$, where $E_0 < E_1 < \cdots < E_{2n}$ are the non-degenerate energies corresponding to the $2n + 1$ *singlet* band-edge states $\Psi_i(x)$, $H\Psi_i = E_i\Psi_i$, $i = 0, 1, \ldots, 2n$. Since parity operator R is an integral, each singlet state $\Psi_i(x)$ has a definite parity, $+1$ or -1. The energy levels in the interior of permitted bands, $E \in (E_{2i}, E_{2i+1})$, $i = 0, \ldots, n$, are doubly degenerate, and certain linear combinations of corresponding Bloch–Floquet doublet states are the eigenstates of R with eigenvalues $+1$ and -1. These properties indicate on the presence of a *hidden bosonized* $N = 2$ *supersymmetry* in any finite-gap system, for which operator R has to play the role of the grading operator. The presence of $2n + 1 \geqslant 3$ singlet states indicates, however, on its nonlinear nature [31–33]. The supercharges and the form of the corresponding nonlinear superalgebra can easily be identified.

[2] Here and in what follows we do not count the prohibited band $(-\infty, E_0)$ that always presents in any periodic system with a smooth potential.

[3] In this sense, and in a contrast with, for example, the Kronig–Penney model, finite-gap potentials play the same role in solid-state physics as the Kepler problem in atomic theory.

Any finite-gap system is characterized by the presence of a nontrivial integral of motion in the form of an anti-Hermitian differential operator of order $2n+1$,

$$A_{2n+1} = D^{2n+1} + c_2^A(x)D^{2n-1} + c_3^A(x)D^{2n-2} + \cdots + c_{2n}^A(x), \tag{2.10}$$

where the coefficient functions $c_i^A(x)$ are real. The absence of the term proportional to D^{2n} in its structure, i.e. an equality $c_1^A(x) = 0$, is dictated by the condition $[A_{2n+1}, H] = 0$. Other coefficients $c_j^A(x)$ are fixed in the form of polynomials in the potential $u(x)$ and its derivatives [5]. Thus, for periodic potential, A_{2n+1} is a periodic operator, i.e. $[A_{2n+1}, T] = 0$. (A_{2n+1}, H) is known as the Lax pair of the nth-order Korteweg–de Vries (KdV) equation. A possible form of the n-gap potential is fixed by a nonlinear equation, which has a sense of the nth equation of the stationary KdV hierarchy [2, 5]. This equation can be represented alternatively as

$$\tilde{L}(J\tilde{L})^n 1 = 0, \qquad \tilde{L} = D^3 + 2(uD + Du), \tag{2.11}$$

where J is the operator of indefinite integration $J = D^{-1}$ [4]. The form of a *one-gap* potential is fixed by this equation in a unique manner, $u(x) = 2\mathcal{P}(x + \omega_2 + c)$, where $\mathcal{P}(x)$ is the doubly periodic (elliptic) Weierstrass function [36], and c is a constant. To have a real-valued potential, one of the periods of $\mathcal{P}(x)$ is chosen to be real, $2\omega_1 = 2L$, while another period $2\omega_2$ is assumed to be pure imaginary, and $c \in \mathbb{R}$. In the case $n > 1$, the form of the potential $u(x)$ is not fixed uniquely even if it is restricted to a class of elliptic functions.

The mutually commuting operators A_{2n+1} and H satisfy the relation

$$-A_{2n+1}^2 = P_{2n+1}(H), \qquad P_{2n+1}(H) = \prod_{j=0}^{2n}(H - E_j), \tag{2.12}$$

where $P_{2n+1}(H)$ is a spectral polynomial given in terms of singlet energies. It is in accordance with Burchnall–Chaundy theorem [37, 38], which says that if two differential in x operators A and B of mutually prime orders l and k, respectively, commute, $[A, B] = 0$, they satisfy a relation $P(A, B) = 0$, where P is a polynomial of order k in A, and of order l in B. Equation (2.12) corresponds to a non-degenerate ($E_i \neq E_j$ for $i \neq j$) spectral elliptic curve of genus n associated with an n-gap periodic system [1–4][4].

As a consequence of (2.12), the operator A_{2n+1} annihilates all the $2n+1$ singlet band-edge states. Indeed, from $[A_{2n+1}, H] = 0$ we have $A_{2n+1}\Psi_j = \alpha\Psi_j + \beta\Phi_j$, where Ψ_j is a physical ($T\Psi_j = \gamma\Psi_j, \gamma \in \{-1, 1\}$) and Φ_j is a non-physical solution corresponding to a band-edge energy E_j. Acting from the left by T, we get $\gamma A_{2n+1}\Psi_j = \gamma\alpha\Psi_j + \beta T\Phi_j$, and, therefore, $\beta(\gamma T - 1)\Phi_j = 0$. As Φ_j is neither periodic nor antiperiodic, the last equation can be satisfied if and only if $\beta = 0$. Then, equation (2.12) dictates that $\alpha = 0$.

Consider the Wronskian of the singlet states, $W^A \equiv W(\Psi_0, \ldots, \Psi_{2n})$. In a generic case the Wronskian of s linearly independent functions that form a kernel of an arbitrary linear differential operator of order s, $\mathcal{L} = D^s + c_1(x)D^{s-1} + \cdots$, satisfies the Abel identity $W'(x) = -c_1(x)W$ [38]. For operator (2.10) a corresponding coefficient function is $c_1^A(x) = 0$, and because of the linear independence of the singlet band-edge states we find that

$$W^A(x) = C \neq 0, \tag{2.13}$$

where C is a constant. When s linearly independent zero modes φ_j, $j = 1, \ldots, s$, of operator $\mathcal{L}$ are known, the form of this operator can be reconstructed in their terms. The coefficients $c_k(x)$ are defined by relations $c_k(x) = -\frac{W_k}{W}$, $k = 1, \ldots, s$, where the functions $W_k(x)$ are obtained from Wronskian $W = W(\varphi_1, \ldots, \varphi_s)$ by replacing in it $\varphi_j^{(s-k)} \equiv D^{s-k}\varphi_j$ by $\varphi_j^{(s)}$ [40] , see appendix A. In our case, each singlet band-edge state $\Psi_i(x)$, being a zero mode of

[4] Because of the described properties, $u(x)$ is called algebro-geometric finite-gap potential.

A_{2n+1}, possesses a definite parity. As a result, with taking into account (2.13), we find that the coefficients $c^A_{2r}(x)$ are *odd*, while the coefficients $c^A_{2r+1}(x)$ are *even* non-singular functions. Hence the integral A_{2n+1} is parity odd,

$$\{R, A_{2n+1}\} = 0. \tag{2.14}$$

Introducing two Hermitian operators

$$Z = Z_1 = \mathrm{i}A_{2n+1}, \qquad Z_2 = \mathrm{i}RZ, \tag{2.15}$$

and identifying them as odd supercharges, we conclude that any finite-gap periodic system with even smooth potential is characterized by a hidden bosonized nonlinear $N = 2$ supersymmetry of order $2n + 1$ [32, 33, 39],

$$\{Z_a, Z_b\} = 2\delta_{ab} P_{2n+1}(H), \qquad a, b = 1, 2. \tag{2.16}$$

3. Tri-supersymmetric extensions of finite-gap systems

In this section, we show that the application of a non-singular Crum–Darboux transformation to a finite-gap system produces a partner system with identical spectrum, and study a peculiar supersymmetry appearing in the obtained isospectral pair.

3.1. Darboux transformations and supersymmetry

A usual model of supersymmetric quantum mechanics is based on a Darboux transformation, by which an (almost) isospectral system can be associated with a given quantum system.

Consider a Hamiltonian $H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x)$, and an eigenstate $\psi_\star$ corresponding to a fixed eigenvalue $E_\star$, $H\psi_\star = E_\star\psi_\star$. Here we do not assume any regularity conditions for $\psi_\star$. It can be a physical eigenstate, or a second, non-physical solution of the second-order differential equation, corresponding to a physical energy level $E_\star$, or can be a solution corresponding to a nonphysical value $E_\star$. The Darboux transformation is generated by a first-order differential operator $A_1 = \frac{\mathrm{d}}{\mathrm{d}x} - (\ln\psi_\star)'$, which annihilates $\psi_\star$, $A_1\psi_\star = 0$, and relates H with another Hamiltonian

$$\tilde{H} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tilde{u}(x), \qquad \tilde{u}(x) = u(x) - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\ln\psi_\star, \tag{3.1}$$

by means of an intertwining relation

$$A_1 H = \tilde{H} A_1. \tag{3.2}$$

Then for eigenstates of two Hamiltonians corresponding to the same arbitrary value of energy E, we have

$$H\psi_E = E\psi_E, \qquad \tilde{H}\tilde{\psi}_E = E\tilde{\psi}_E, \tag{3.3}$$

$$\tilde{\psi}_E = \frac{1}{\sqrt{E - E_\star}} A_1\psi_E, \qquad \psi_E = \frac{1}{\sqrt{E - E_\star}} A_1^\dagger\tilde{\psi}_E. \tag{3.4}$$

Relations (3.3) and (3.4) have a symmetry $H \leftrightarrow \tilde{H}$, $\psi_E \leftrightarrow \tilde{\psi}_E$, $A \leftrightarrow A^\dagger$. This reflects a property that the transformation corresponding to the adjoint intertwining relation

$$A_1^\dagger\tilde{H} = H A_1^\dagger \tag{3.5}$$

is generated by the operator $A_1^\dagger$, which annihilates a state $\tilde{\psi}_\star = 1/\psi_\star$, $A_1^\dagger\tilde{\psi}_\star = 0$, and acts in an opposite direction by relating $\tilde{H}$ with H. It is easy to see that the both Hamiltonians can be represented in terms of operators A_1 and $A_1^\dagger$, $H = A_1^\dagger A_1 + E_\star$, $\tilde{H} = A_1 A_1^\dagger + E_\star$.

Usually, the Darboux transformation is chosen to annihilate a nodeless physical ground state ψ_0 with energy E_0. In such a case, the potentials $u(x)$ and $\tilde{u}(x)$ are both smooth and regular, or both have the same singularities[5]. In a *non-periodic* case, the physical nodeless ground state ψ_0 vanishes at the ends of a (possibly infinite) interval. As a consequence, there is no physical partner state with the same energy in the spectrum of $\tilde{H}$. Indeed, the state $\tilde{\psi}_0 = 1/\psi_0$ annihilated by $A_1^\dagger$ is divergent at infinity and is not physical. In this case both systems are almost isospectral, their spectrum is the same except the energy level E_0 to be absent from the spectrum of $\tilde{H}$. Note that from the viewpoint of the adjoint intertwining relation (3.5), the transformation from $\tilde{H}$ to H is generated by the operator $A_1^\dagger$ associated with a nonphysical state $\tilde{\psi}_0 = 1/\psi_0$, which corresponds to a nonphysical for $\tilde{H}$ eigenvalue E_0. On the other hand, in correspondence with (3.4), for $E = E_0$ we still have relations $\psi_0 = A_1^\dagger \tilde{\eta}_0$ and $\tilde{\psi}_0 = 1/\psi_0 = A_1 \eta_0$, but $\tilde{\eta}_0 = -\frac{1}{\psi_0}\int^x \psi_0^2(x)\,\mathrm{d}x$, $\eta_0 = \psi_0 \int^x \psi_0^{-2}\,\mathrm{d}x$ are the non-physical, non-normalizable solutions of the equations $H\eta_0 = E_0\eta_0$ and $\tilde{H}\tilde{\eta}_0 = E_0\tilde{\eta}_0$.

In the periodic case with ψ_0 corresponding to the singlet band-edge state of the lowest energy, η_0 and $\tilde{\eta}_0$ are the non-physical, non-periodic divergent solutions. From this discussion it is also clear that if the Darboux transformation is realized with a nodeless state $\psi_\star$ such that both states $\psi_\star$ and $1/\psi_\star$ are not physical (non-normalizable), the energy level $E_\star$ is absent from the spectra of both partner systems, and physical energy levels satisfy a relation $E > E_\star$.

The relation between the Darboux transformation and the usual supersymmetric quantum mechanics is direct. The Hamiltonians H and $\tilde{H}$ shifted for the constant $E_\star$ are known as superpartner Hamiltonians, and form a superextended system described by the matrix Hamiltonian

$$H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \tag{3.6}$$

where

$$H_- \equiv H - E_\star = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2 - W', \qquad H_+ \equiv \tilde{H} - E_\star = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2 + W', \tag{3.7}$$

and $W(x)$ is a superpotential, $W = -\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_\star$.

With the Darboux transformation, two Hermitian linear differential matrix operators

$$Q_1 = \begin{pmatrix} 0 & A_1 \\ A_1^\dagger & 0 \end{pmatrix}, \qquad Q_2 = \mathrm{i}\sigma_3 Q_1 \tag{3.8}$$

are associated, in terms of which intertwining relations (3.2) and (3.5) take a form of conservation laws for supercharges Q_a, $[Q_a, H] = 0$, $a = 1, 2$. Together with Hamiltonian (3.6) they generate the linear $N = 2$ superalgebra

$$[Q_a, H] = 0, \qquad \{Q_a, Q_b\} = 2\delta_{ab}H. \tag{3.9}$$

A diagonal Pauli matrix σ_3 plays here the role of the grading operator, $[\sigma_3, H] = 0$, $\{\sigma_3, Q_a\} = 0$.

In a non-periodic case, if $\psi_\star$ or $1/\psi_\star$ is normalizable, there exists a two-component physical state annihilated by both matrix supercharges, which is a ground state of zero energy of one of the superpartner subsystems. It is invariant under corresponding supersymmetry transformations generated by Q_a, and we have the case of exact, unbroken supersymmetry. The supersymmetric doublets of states corresponding to positive energies are mutually transformed by supercharges Q_a in correspondence with (3.4). In the case if both $\psi_\star$ and $1/\psi_\star$ are not physical, all the states of supersymmetric system (3.6) are organized in supersymmetric

[5] Singular Darboux transformations generated by the states with nodes also find some applications, see [41].

doublets, including the states of the lowest energy, that takes here a nonzero, positive value. This picture corresponds to the broken supersymmetry, which describes a pair of completely isospectral superpartner systems.

In the case of a periodic quantum system with a smooth potential, the supersymmetric system (3.6) constructed on the base of the Darboux transformation with a nodeless singlet band-edge state Ψ_0 will be characterized by zero-energy doublet of the states given by the columns $(0, \Psi_0)^t$ and $(1/\Psi_0, 0)^t$. Both these states are annihilated by the supercharges Q_a, and the corresponding $N = 2$ supersymmetry is unbroken. Here the superpartner systems are completely isospectral as in the non-periodic case with broken supersymmetry.

3.2. Higher-order Crum–Darboux transformations

(Almost) isospectral systems can also be related by differential operators of higher order, which corresponds to the situation well described by the generalization of the Darboux transformation due to Crum.

Let a differential operator A_k of order k, $A_k = D^k + \sum_{j=1}^{k} c_j^A D^{k-j}$, annihilates a space V spanned by k eigenstates of the Hamiltonian H, $V = \mathrm{span}\{\psi_1, \ldots, \psi_k\}$, which are not obligatorily physical. Then, there holds the relation

$$A_k H = \tilde{H} A_k, \qquad \tilde{H} = H + 2\big(c_1^A\big)' = H - 2(\ln W(\psi_1, \ldots, \psi_k))''. \quad (3.10)$$

In the case of the Darboux transformation ($k = 1$), the Wronskian of a single function is the function itself, and (3.10) reduces to the intertwining relation of the standard supersymmetry. The operators A_k and $A_k^\dagger$ produce the relations of the form (3.4) for energies $E \neq E_i, i = 1, \ldots, k$. For $k > 1$, equation (3.10) underlies a higher-order (nonlinear) generalization of supersymmetric quantum mechanics, see [31–33]. In a generic case the spectra of H and $\tilde{H}$ are almost identical, their spectra can be different in k or less number of physical eigenvalues. For a quantum system described by H, one can obtain various partner Hamiltonians $\tilde{H}$, by choosing different sets of eigenstates $\psi_1, \ldots, \psi_k$. However, if we want to get the associated partner Hamiltonian $\tilde{H}$ with the same regularity properties as the initial Hamiltonian H, these states have to be chosen in a special way.

The higher-order Crum–Darboux transformations can be factorized into the consecutive chain of the first-order Darboux, or the second-order Crum transformations, see [42]. We are interested in the conditions which not only ensure the regularity of the transformations for a periodic even finite-gap system, but also produce an isospectral *even* partner potential. In a periodic system, a regular transformation of the second order can be obtained if the kernel of the operator A_2 consists of the states corresponding to the edges of the same prohibited band [43]. These two states, due to the oscillation theorem mentioned in the previous section, have the same period and the same number of alternating nodes. The Wronskian of the functions selected in this way is a function of a definite sign, not taking the zero value. Consider a Crum–Darboux transformation that annihilates the indicated pairs of the edge states. In the case if the order of the transformation is odd, it has also to annihilate the nodeless ground state Ψ_0. This guarantees a smooth and singularity-free potential of the partner Hamiltonian.

Concluding, we can construct a whole family of the partner finite-gap periodic systems by means of the Crum–Darboux transformations, just by choosing appropriately the singlet states of the original system, respecting the rules described above. For instance, the generator of a hidden bosonized supersymmetry Z produces a Crum–Darboux transformation associated with the *trivial* selection: it annihilates all the singlets. Since the Wronskian of all the singlet states is a nontrivial constant, see equation (2.13), we find that the partner Hamiltonian coincides with the original one, and the intertwining relation reduces to the relation of commutation of Z and H.

Below, by means of nontrivial Crum–Darboux transformations, we shall construct a family of $2^n - 1$ different completely isospectral partner systems for a given arbitrary n-gap periodic system, and reveal a special nonlinear supersymmetry appearing in any pair of the total family of 2^n systems. The key role in the construction will belong to the already described hidden bosonized supersymmetry.

It is worth noting here that a regular Crum–Darboux transformation for finite-gap periodic systems can also be produced by making use of certain Bloch functions [28, 29, 44–46]. A partner for parity-even potential obtained in such a way in a generic case, however, will not be an even function. We shall return to this point in the last section.

3.3. Tri-supersymmetric extensions and centrally extended nonlinear $N = 4$ supersymmetry

Consider an n-gap periodic system, and mark $r \leqslant n$ prohibited bands in its spectrum. $2r$ singlet physical states at the edges of these prohibited bands span a $2r$-dimensional linear vector space which we denote by V_+. The Wronskian of the corresponding $2r$ singlet band-edge states is a nodeless $2L$-periodic even function. Let Q_+ be a linear differential operator of order $2r$ that annihilates the space V_+,

$$Q_+ = D^{2r} + \sum_{j=1}^{2r} c_j^+(x) D^{2r-j}, \qquad Q_+ V_+ = 0. \tag{3.11}$$

Singlet band-edge states are periodic or anti-periodic, and can be presented by real wavefunctions. So, the coefficient functions in (3.11) are real. Taking also into account that any band-edge state has a definite parity, one can show that (3.11) is a $2L$-periodic even operator, $[T, Q_+] = [R, Q_+] = 0$, see appendix A.

The kernel of the integral Z has a form $\operatorname{Ker} Z = V_+ \oplus V_-$, where V_- is a supplementary $(2(n-r)+1)$-dimensional linear vector space spanned by the rest of singlet band-edge states. Then Z can be decomposed as $Z = S^\dagger Q_+$, where $S^\dagger$ is a differential operator of order $2(n-r)+1$ with the property $S^\dagger Q_+ V_- = 0$. Hermiticity of Z and equations (2.10), (2.15) and (3.11) mean that

$$Z = S^\dagger Q_+ = Q_+^\dagger S, \tag{3.12}$$

and

$$-\mathrm{i}S = D^{2(n-r)+1} + \sum_{j=1}^{2(n-r)+1} c_j^S(x) D^{2(n-r)+1-j}, \tag{3.13}$$

where the coefficient functions are real and $c_1^S(x) = c_1^+(x)$. From the properties of Z and Q_+, we also find that S is a $2L$-periodic parity-odd operator.

Now we show that $\operatorname{Ker} S = V_-$. To this end we note that in accordance with equation (3.10) and the Abel identity $W' = -c_1(x)W$, the equality $c_1^S(x) = c_1^+(x)$ obtained directly from (3.12) means that the application of the Crum–Darboux transformations with the operators Q_+ and S produces the same non-singular superpartner Hamiltonian $\tilde{H} = H + 2\left(c_1^+\right)'$ satisfying the intertwining relations

$$Q_+ H = \tilde{H} Q_+, \qquad S H = \tilde{H} S, \tag{3.14}$$

$$Q_+^\dagger \tilde{H} = H Q_+^\dagger, \qquad S^\dagger \tilde{H} = H S^\dagger. \tag{3.15}$$

The Hamiltonian $\tilde{H}$ describes a periodic system with an even potential of the period $2L$ with n gaps in the spectrum. Thus, there exists an odd Hermitian differential operator $\tilde{Z}$

of the form (2.10) commuting with $\tilde{H}$. The intertwining relations (3.14), (3.15) provide an alternative, two-term decomposed form for $\tilde{Z}$. Indeed, we get $\left[\tilde{H}, SQ_+^\dagger\right] = [\tilde{H}, Q_+S^\dagger] = 0$. Both operators $SQ_+^\dagger$ and $Q_+S^\dagger$ are of order $2n+1$, and should coincide with $\tilde{Z}$ up to some polynomial in $\tilde{H}$. However, they anticommute with reflection operator R, and this implies that

$$\tilde{Z} = SQ_+^\dagger = Q_+S^\dagger. \tag{3.16}$$

Take any of $2(n-r)+1$ singlet states $\Psi_- \in V_-$ not annihilated by Q_+. Multiplying the equation $Z\Psi_- = 0$ by Q_+ from the left and using (3.12) and (3.16), we get $SQ_+^\dagger Q_+\Psi_- = 0$. The operator $Q_+^\dagger Q_+$ is $2L$-periodic operator, and as it follows from (3.14) and (3.15), commutes with the Hamiltonian H. It changes neither energy nor the period of a singlet state Ψ_-, and then $Q_+^\dagger Q_+\Psi_- = \alpha\Psi_-$, where α is a nonzero number. Hence, $SQ_+^\dagger Q_+\Psi_- = \alpha S\Psi_- = 0$, and we conclude that $\mathrm{Ker}\, S = V_-$. Changing the notation, $Q_- \equiv S$, we have

$$Z = Q_+^\dagger Q_- = Q_-^\dagger Q_+, \qquad \tilde{Z} = Q_+Q_-^\dagger = Q_-Q_+^\dagger, \tag{3.17}$$

and

$$\mathrm{Ker}\, Q_+ \oplus \mathrm{Ker}\, Q_- = \mathrm{Ker}\, Z, \qquad \mathrm{Ker}\, Q_+^\dagger \oplus \mathrm{Ker}\, Q_-^\dagger = \mathrm{Ker}\, \tilde{Z}. \tag{3.18}$$

This result means a complete isospectrality of the finite-gap periodic systems described by the Hamiltonians H and $\tilde{H}$. Indeed, in accordance with the properties of the Crum–Darboux transformations, the action of both operators Q_+ and Q_- on any doublet of eigenstates of H from the interior of permitted bands transforms it into a doublet of eigenstates of $\tilde{H}$ with the same energy value. The adjoint operators $Q_+^\dagger$ and $Q_-^\dagger$ act in the same way in the opposite direction. The singlet states of H annihilated by Q_+ (or Q_-) are transformed by Q_- (or Q_+) into zero modes of $Q_+^\dagger$ (or $Q_-^\dagger$) being the singlet states of $\tilde{H}$ of the same energy. The same picture is valid for the singlet states of $\tilde{H}$ annihilated by $Q_+^\dagger$ (or $Q_-^\dagger$) and transformed by Q_- (or Q_+) into the corresponding singlet states of H.

The intertwining relations (3.14) and (3.15) as well as factorization of the supercharges of bosonized supersymmetry can be rewritten in a compact form once we use the matrix formalism and define an extended Hamiltonian $\mathcal{H}$ and operators $\mathcal{Q}_\pm$ and $\mathcal{Z}$,

$$\mathcal{H} = \begin{pmatrix} \tilde{H} & 0 \\ 0 & H \end{pmatrix}, \qquad \mathcal{Q}_\pm = \begin{pmatrix} 0 & Q_\pm \\ Q_\pm^\dagger & 0 \end{pmatrix}, \qquad \mathcal{Z} = \begin{pmatrix} \tilde{Z} & 0 \\ 0 & Z \end{pmatrix}. \tag{3.19}$$

Here relations (3.14), (3.15) and (3.17) can be presented as

$$[\mathcal{H}, \mathcal{Z}] = 0, \qquad [\mathcal{H}, \mathcal{Q}_\pm] = 0, \tag{3.20}$$

$$\mathcal{Z} = \mathcal{Q}_-\mathcal{Q}_+ = \mathcal{Q}_+\mathcal{Q}_-. \tag{3.21}$$

The triplet $\mathcal{Q}_+$, $\mathcal{Q}_-$ and $\mathcal{Z}$ is a set of *commuting* integrals for the superextended system described by the matrix Hamiltonian $\mathcal{H}$. There exists a common basis, in which $\mathcal{Q}_\pm$, $\mathcal{Z}$ and $\mathcal{H}$ are diagonal, and since all these operators are self-adjoint, their eigenvalues are real. We have a chain of equalities

$$\mathcal{Z}^2 = \mathcal{Q}_+\mathcal{Q}_-\mathcal{Q}_+\mathcal{Q}_- = \mathcal{Q}_+^2\mathcal{Q}_-^2 = P_{\mathcal{Z}}(\mathcal{H}) = \prod_{j=1}^{2n+1}(\mathcal{H} - E_j), \tag{3.22}$$

where $P_{\mathcal{Z}}$ is a positive-semidefinite spectral polynomial, and E_j are energies of the singlet states of subsystems. In correspondence with (3.22), the band-edge states of the extended system are organized in supersymmetric doublets, on which $\mathcal{Z}$ takes zero values. The states of the interior of permitted bands are organized in energy quadruplets, on certain pairs of which $\mathcal{Z}$ takes nonzero values $\pm\sqrt{P_{\mathcal{Z}}(E)}$. The diagonal components of $\mathcal{Q}_\pm^2$ consist of differential

operators of orders $4r$ and $4(n-r)+2$. One of these two numbers is less than $2n+1$. Suppose that it is the case of the operator $\mathcal{Q}_+^2$. Its lower diagonal component $Q_+^\dagger Q_+$ satisfies a relation $[H, Q_+^\dagger Q_+] = 0$. According to the general theory of finite-gap systems, the only operators commuting with H of order lower then $2n+1$ are polynomials in the Hamiltonian, and we conclude that $Q_+^\dagger Q_+$ is such a polynomial, which has to take zero values on $2r$ singlets belonging to Ker Q_+. The same arguments hold for the upper component of the operator $\mathcal{Q}_+^2$, and we find that

$$\mathcal{Q}_+^2 = P_+(\mathcal{H}) = \prod_{j=1}^{2r} \left(\mathcal{H} - E_j^+\right). \tag{3.23}$$

With making use of (3.22) and (3.23), we get also

$$\mathcal{Q}_-^2 = P_-(\mathcal{H}) = \prod_{j=1}^{2(n-r)+1} \left(\mathcal{H} - E_j^-\right), \tag{3.24}$$

where $P_\pm(\mathcal{H})$ are positive-semidefinite operators, and $E^\pm$ are the energies of the corresponding band-edge states annihilated by $\mathcal{Q}_\pm$. The eigenvalues of $\mathcal{Q}_\pm$ are $\pm\sqrt{P_\pm(E)}$, where the signs of square roots are correlated with the square root sign of corresponding eigenvalue of $\mathcal{Z}$ in accordance with equation (3.21).

In addition to the nontrivial integrals of motion $\mathcal{Z}$ and $\mathcal{Q}_\pm$, the Hamiltonian $\mathcal{H}$ possesses another triplet of trivial, mutually commuting, integrals

$$\Gamma_1 = \sigma_3, \qquad \Gamma_2 = R, \qquad \Gamma_3 = R\sigma_3, \tag{3.25}$$

which satisfy the relations $\Gamma_i^2 = 1, i = 1, 2, 3, \Gamma_1\Gamma_2\Gamma_3 = 1$. Any of Γ_i can be chosen as a $\mathbb{Z}_2$-grading operator Γ_*. Any of the nontrivial integrals of motion either commutes or anti-commutes with any of the trivial integrals. Fixing the grading operator, we classify any nontrivial integral as a bosonic or fermionic, while the Hamiltonian and the trivial integrals (3.25) are always identified as bosonic operators. In correspondence with this identification, a certain superalgebra is generated. For all the three possible choices of the grading operator, one of the nontrivial integrals plays the role of a bosonic, $\mathbb{Z}_2$-even operator, while the other two integrals are classified as fermionic, $\mathbb{Z}_2$-odd operators, see the table below. We name this structure the *tri-supersymmetry*[6].

Grading operator	σ_3	R	$\sigma_3 R$
Bosonic integral	$\mathcal{Z}$	$\mathcal{Q}_+$	$\mathcal{Q}_-$
Fermionic integrals	$\mathcal{Q}_+, \mathcal{Q}_-$	$\mathcal{Z}, \mathcal{Q}_-$	$\mathcal{Z}, \mathcal{Q}_+$

The complete structure of the tri-supersymmetry will be described in the following subsection. Here, let us choose $\Gamma_* = \sigma_3$ as the grading operator, and discuss the corresponding nonlinear supersymmetric subalgebra generated by the *local* integrals of motion, forgetting for the moment the nonlocal integral R. Introduce the notation

$$\mathcal{Q}_\pm^{(1)} = \mathcal{Q}_\pm, \qquad \mathcal{Q}_\pm^{(2)} = \mathrm{i}\sigma_3\mathcal{Q}_\pm. \tag{3.26}$$

These fermionic supercharges together with bosonic operators $\mathcal{Z}$ and $\mathcal{H}$ generate the superalgebra

$$\left\{\mathcal{Q}_+^{(a)}, \mathcal{Q}_+^{(b)}\right\} = 2\delta^{ab} P_+(\mathcal{H}), \qquad \left\{\mathcal{Q}_-^{(a)}, \mathcal{Q}_-^{(b)}\right\} = 2\delta^{ab} P_-(\mathcal{H}), \tag{3.27}$$

[6] Such a structure was observed for the first time in the $N = 2$ superextended Dirac delta potential problem [47], where the basic triplet of nontrivial integrals has a completely different nature. In particular, there both supercharges of the hidden bosonized $N = 2$ linear supersymmetry of the form (3.9) are nonlocal operators, cf (2.16) and (2.15)

$$\left\{\mathcal{Q}_{+}^{(a)}, \mathcal{Q}_{-}^{(b)}\right\} = 2\delta^{ab}\mathcal{Z}, \tag{3.28}$$

$$\left[\mathcal{H}, \mathcal{Q}_{\pm}^{(a)}\right] = [\mathcal{H}, \mathcal{Z}] = \left[\mathcal{Z}, \mathcal{Q}_{\pm}^{(a)}\right] = 0. \tag{3.29}$$

Superalgebra (3.27), (3.28) and (3.29) is identified as a centrally extended *nonlinear* $N = 4$ supersymmetry, in which $\mathcal{Z}$ plays a role of the bosonic central charge[7].

The supercharges $\mathcal{Q}_{+}^{(a)}$ annihilate a part of the band-edge states organized in supersymmetric doublets, while another part of supersymmetric doublets is annihilated by the supercharges $\mathcal{Q}_{-}^{(a)}$. The band-edge states which do not belong to the kernel of the supercharges $\mathcal{Q}_{+}^{(a)}$ (or $\mathcal{Q}_{-}^{(a)}$) are transformed (rotated) by these supercharges within the corresponding supersymmetric doublet. The bosonic central charge $\mathcal{Z}$ annihilates all the band-edge states. So, we have here the picture reminiscent somehow of the partial supersymmetry breaking appearing in supersymmetric field theories with BPS-monopoles [49].

3.4. Tri-supersymmetry and su(2|2)

Let us study the complete algebraic structure of the tri-supersymmetry. To do this, consider the set of integrals of motion created by the multiplicative combinations of the trivial integrals with nontrivial ones,

$$\mathcal{H}, \qquad \Gamma_i, \qquad \Gamma_\alpha\mathcal{Z}, \qquad \Gamma_\alpha\mathcal{Q}_{+}, \qquad \Gamma_\alpha\mathcal{Q}_{-}, \tag{3.30}$$

where $\alpha = 0, 1, 2, 3$, and by Γ_0 we denote a unit two-dimensional matrix. Each of these integrals either commutes or anticommutes with any of Γ_i defined in (3.25). Identifying one of Γ_i as the $\mathbb{Z}_2$-grading operator Γ_*, we separate the set (3.30) into eight $\mathbb{Z}_2$-even (bosonic) operators commuting with Γ_*, and eight $\mathbb{Z}_2$-odd (fermionic) operators, which anticommute with Γ_*. Though this separation depends on the choice of Γ_*, the superalgebra in all three cases has, in fact, the same structure.

To reveal this common superalgebraic structure for all the three possible choices of the grading operator, we denote the fermionic operators as $(F_1, \ldots, F_8)$. The set of bosonic operators we write as $(\mathcal{H}, \Gamma_*, \Sigma_1, \Sigma_2, B_1, \ldots, B_4)$, where $\Sigma_{1,2}$ are two trivial integrals from the set (3.25) to be different from Γ_*. Denote by $P_B(\mathcal{H})$ a universal polynomial produced by the square of any of the four integrals B_a, $B_a^2 = P_B(\mathcal{H}), a = 1, \ldots, 4$. Given Γ_*, we can separate the set of fermionic operators in two subsets depending on the commutation relations with the integrals Σ_1 and Σ_2. The subset which commutes with Σ_2, we label as $F_\mu, \mu = 1, \ldots, 4, [\Sigma_2, F_\mu] = 0$, and denote a universal polynomial corresponding to a square of any of these fermionic operators by P_{22}, $F_\mu^2 = P_{22}$. For the subset of fermionic operators which commute with Σ_1, we put the index $F_\lambda, \lambda = 5, \ldots, 8, [\Sigma_1, F_\lambda] = 0$, and denote the analogous universal polynomial by P_{11}, $F_\lambda^2 = P_{11}$. Identifying in the described way the integrals F_μ, F_λ, B_a, Σ_1 and Σ_2, and computing directly all the (anti)commutators, we find the superalgebra of tri-supersymmetry, which may be presented in the same form modulo some special polynomials in dependence on the choice of Γ_*. These additional polynomials we denote by P_{12}, P_{1B} and P_{2B}, where the subindex with two entries indicates the origin of the (anti)commutation relation. Polynomial P_{12} comes from the anticommutators of F_λ with F_μ, polynomial P_{2B} comes from the commutators between the integrals F_μ, ($[\Sigma_2, F_\mu] = 0$) and

[7] The basic structure of the algebra of finite-order differential operators of a form more general than (3.27)–(3.29) was discussed by Andrianov and Sokolov [48], but outside the context of finite-gap periodic systems and parity-even potentials. In comparison with (3.27)–(3.29), the superalgebraic structure of [48] includes some additional independent polynomial of the Hamiltonian. As a consequence, instead of the relation $\mathcal{Z}^2 = P_+(\mathcal{H})P_-(\mathcal{H})$, which follows from equations (3.22)–(3.24) and reflects the nature and peculiarities of the band structure [34], its analog in [48] has a different form, see equations (40), (41), (43) and (46) there.

B_a, polynomial P_{1B} has analogous sense. Since the polynomials P_{12}, P_{1B} and P_{2B} depend on the grading, their explicit form together with explicit form of bosonic and fermionic operators for all three choices of the grading operator are presented in appendix B, see tables 1–3. With the described notations, the anti-commutation relations for fermionic operators, and commutation relations between bosonic and fermionic operators are presented in tables 4 and 5.

The identification of the complete superalgebra of the tri-supersymmetry can be achieved now if we analyze the still missing commutation relations between $\mathbb{Z}_2$-even generators. Introduce the following linear combinations of them,

$$\mathcal{G}_1^{(\pm)} = \tfrac{1}{4}(B_1 \pm B_3) = \tfrac{1}{2}B_1\Pi_\pm, \qquad \mathcal{G}_2^{(\pm)} = -\tfrac{1}{4}(B_2 \pm B_4) = -\tfrac{1}{2}B_2\Pi_\pm, \tag{3.31}$$

$$J_3^{(\pm)} = \tfrac{1}{4}(\Sigma_1 \pm \Sigma_2) = \tfrac{1}{2}\Sigma_1\Pi_\pm, \tag{3.32}$$

where $\Pi_\pm = \frac{1}{2}(1 \pm \Gamma_*)$ are the projectors. These operators satisfy the following algebra:

$$\left[\mathcal{G}_1^{(\pm)}, \mathcal{G}_2^{(\pm)}\right] = \mathrm{i}J_3^{(\pm)}P_B(\mathcal{H}), \tag{3.33}$$

$$\left[J_3^{(\pm)}, \mathcal{G}_a^{(\pm)}\right] = \mathrm{i}\epsilon_{ab}\mathcal{G}_b^{(\pm)}, \qquad a, b = 1, 2, \tag{3.34}$$

$$\left[\mathcal{G}_a^{(+)}, \mathcal{G}_b^{(-)}\right] = \left[J_3^{(+)}, \mathcal{G}_a^{(-)}\right] = \left[J_3^{(-)}, \mathcal{G}_a^{(+)}\right] = \left[J_3^{(+)}, J_3^{(-)}\right] = 0. \tag{3.35}$$

The commutation relations (3.33)–(3.35) correspond to the direct sum of two deformed $su(2)$ algebras in which $\mathcal{H}$ plays a role of a multiplicative central charge. This bosonic subalgebra is reminiscent of the nonlinear algebra satisfied by the Laplace–Runge–Lenz and angular momentum vectors in the quantum Kepler problem [50, 51].

It is known that in the case of the quantum Kepler problem, its nonlinear symmetry algebra is reduced on the subspaces of fixed energy $E < 0$, $E = 0$ and $E > 0$ to the Lie algebras $so(4)$, $so(3, 1)$ and $e(3)$, respectively, where $e(3)$ is the 3D Euclidean algebra. Let us see what happens with our tri-supersymmetry under similar reduction. First, consider any 4-fold degenerate energy level $E \neq E_i$ corresponding to the interior part of any permitted band. Rescaling the operators, $\mathcal{G}_a^{\pm} \to J_a^{(\pm)} = \mathcal{G}_a^{(\pm)}/P_B(E)$, we find that together with $J_3^{(\pm)}$ they generate the Lie algebra $su(2) \oplus su(2)$. These operators satisfy the relations $J_i^{(+)}J_i^{(+)} = \frac{3}{4}\Pi_+$, $J_i^{(-)}J_i^{(-)} = \frac{3}{4}\Pi_-$, where the summation in $i = 1, 2, 3$ is assumed. The two common eigenstates of the Hamiltonian, with energy $E \neq E_i$, and of the grading operator Γ_*, with eigenvalue +1 or −1, carry the $1/2 \oplus 0$, or $0 \oplus 1/2$ representations of $su(2) \oplus su(2)$, where the first (second) term corresponds to the generators $J_i^{(+)}$ $\left(J_i^{(-)}\right)$. The fermionic generators mutually transform the states from the two eigenspaces of the grading operator. In accordance with the total number of independent fermionic generators, the energy subspace with $E \neq E_i$ carries an irreducible representation of the $su(2|2)$ superunitary symmetry, which is a supersymmetric extension of the bosonic symmetry $u(1) \oplus su(2) \oplus su(2)$, where the $u(1)$ subalgebra is generated by the grading operator, see [52]. Having in mind that the Hamiltonian appears in a generic form of the superalgebra as a multiplicative central charge, we conclude that the system possesses a nonlinear $su(2|2)$ superunitary symmetry in the sense of [31, 32, 51].

If we reduce our extended system to the subspace corresponding to any doubly degenerate energy level E_i corresponding to a doublet of band-edge states, the bosonic part of the superalgebra is reduced to the algebra $u(1) \oplus e(2) \oplus e(2)$, where the first term corresponds to the integral Γ_*, while other two correspond to the two copies of the 2D Euclidean algebras generated in the eigensubspaces of Γ_* by the rotation operators $J_3^{(\pm)}$ and commuting translation

generators $\mathcal{G}_a^{(\pm)}$. Note that the supersymmetry of the form similar to the present one reduced to a level E_i was analyzed in [53] in the context of spontaneous supersymmetry breaking in 3+1 dimensions.

3.5. Self-isospectrality conjecture

In the realm of supersymmetric quantum mechanics associated with linear superalgebraic structure, the complete isospectrality in the non-periodic systems is related to the supersymmetry breaking, which means that the doublet of the ground states is not annihilated by supercharges. In [25], Dunne and Feinberg considered supersymmetric extensions of periodic potentials. They argued that in contrary to the usual situation, the complete isospectrality of superpartner Hamiltonians could appear without violation of the supersymmetry. As one of the examples of the situation they presented a one-gap Lamé Hamiltonian, where the super-symmetric extension was provided by the first-order Darboux transformation corresponding to the first-order supercharges $\mathcal{Q}_-^{(a)}$ defined by (3.26). The superpartner Hamiltonian showed to be the original one but displaced for a half of the period. As we mentioned at the beginning, such a phenomenon of a half-period displacement of superpartners was named in [25] the self-isospectrality. We explained above how the complete isospectrality emerges due to the tri-supersymmetry, namely its local part (3.27)–(3.29). In this framework, the symmetries of the $N = 2$ superextended one-gap Lamé system have to be completed by adding two other supercharges $\mathcal{Q}_+^{(a)}$ of order 2 and a bosonic integral $\mathcal{Z}$ of order 3. The second-order supercharges $\mathcal{Q}_-^{(a)}$ do not annihilate the doublet of the ground states, and the tri-supersymmetry is spontaneously partially broken. As we showed, this turns out to be a general feature of the tri-supersymmetric systems constructed by extension of a finite-gap periodic system by means of a regular Crum–Darboux transformation.

We could ask, motivated by [25], for the indications on the self-isospectrality in the tri-supersymmetric extensions of the finite-gap systems. In our current setting, the self-isospectrality arises if the translation in the half-period L provokes inversion of the Wronskian,

$$W^{\pm}(x + L) = \mathcal{C}^{\pm} \frac{1}{W^{\pm}(x)}, \tag{3.36}$$

where $\mathcal{C}^{\pm}$ are some nonzero constants. Indeed, such a displacement produces changes in the sign for coefficient functions $c_1^{\pm}(x) = -(\ln W^{\pm})'$ of the operators $Q_{\pm}$, and therefore transforms the latter into their conjugates,

$$c_1^{\pm}(x + L) = -c_1^{\pm}(x), \qquad Q_{\pm}(x + L) = Q_{\pm}^{\dagger}(x). \tag{3.37}$$

Making the translation for L in the intertwining relations and comparing the result with their conjugates, we reveal that

$$\tilde{H}(x) = H(x + L), \tag{3.38}$$

and therefore the self-isospectrality does appear.

The construction considered in this section provides a receipt how to get isospectral tri-super-symmetric partners for a given n-gap Hamiltonian. We have seen how the partner Hamiltonian $\tilde{H}$ is determined uniquely once we make a separation of the singlet states into two disjoint families. There exist $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ distinct separations which respect the rules explained in the subsection on the Darboux–Crum transformations. Since one of them is trivial (includes all the singlet states and corresponding integral Z commutes with the Hamiltonian H) we end up with $2^n - 1$ tri-supersymmetric isospectral extensions of the given n-gap system. All the isospectral extensions can be obtained by successive first-order Darboux and second-order Crum transformations.

If antiperiodic singlet states are present in the spectrum, among the possible separations of the singlet states there exists an exceptional one, given by sorting out the singlets into mutually orthogonal families of periodic and antiperiodic states. Despite the lack of the proof, we conjecture that this 'natural' separation leads to the *self-isospectral supersymmetry* characterized by the partner Hamiltonian $\tilde{H}$ to be the original one but displaced in the half of the period.

Suppose an n-gap system H with $n > 1$ has antiperiodic singlet states in its spectrum, and $\tilde{H}$ is a shifted for the half-period Hamiltonian obtainable by the Darboux–Crum transformation associated with the specified natural separation of the singlets. Let $Q_\pm$ be the generators of the Crum–Darboux transformation associated with a separation of singlets different from the natural one. Then shifted for the half-period operators $\tilde{Q}_\pm$ will generate a corresponding Crum–Darboux transformation for the system $\tilde{H}$. In such a way we obtain a new pair of self-isospectral systems H_Q and $H_{\tilde{Q}}$:

$$\begin{array}{ccc} H & \xrightarrow{Q_\pm} & H_Q \\ \Big\downarrow{\scriptstyle x\to x+L} & & \Big\downarrow{\scriptstyle x\to x+L} \\ \tilde{H} & \xrightarrow{\tilde{Q}_\pm} & H_{\tilde{Q}} \end{array}$$

Including H and $\tilde{H}$, we can get 2^{n-1} distinct pairs of self-isospectral Hamiltonians. Starting with any n-gap system that has nonzero number of antiperiodic singlet states, we would be able to construct 2^{n-1} extended self-isospectral Hamitonians $\mathcal{H}$.

In the following section, our self-isospectrality conjecture will be supported by the study of the tri-supersymmetry of the associated Lamé equation.

4. Associated Lamé equation and its isospectral extensions

We apply here a general theory developed in the previous sections to a broad class of finite-gap systems described by the associated Lamé equation. In particular, we study the isospectral extension based on the natural separation of the singlet states into periodic and anti-periodic ones, and show that it leads to the self-isospectral tri-supersymmetric systems. We provide an explicit form of both the diagonal and non-diagonal integrals of motion of the extended system. The examples of isospectral extensions not possessing a property of self-isospectrality are presented as well.

Associated Lamé equation is a two-parametric second-order differential equation of Fuchsian type with four singularities and doubly-periodic coefficients,

$$-\psi'' - \left(C_m \,\mathrm{dn}^2 x + C_l \frac{k'^2}{\mathrm{dn}^2 x} + E \right) \psi = 0, \tag{4.1}$$

where $C_m = m(m+1), C_l = l(l+1)$ are real numbers and $\mathrm{dn}\, x \equiv \mathrm{dn}(x,k)$ is Jacobi elliptic function with modular parameter $k \in (0,1)$; $k' \in (0,1)$ is a complementary modular parameter, $k'^2 = 1 - k^2$. Lamé equation ($l = 0$ case), obtained originally by separation of the Laplace equation in elliptical coordinates, has been a subject of extensive studies with use of both analytical [54, 55] and algebraical [8] methods. Due to appealing properties of its solutions, equation (4.1) found the applications in diverse areas of physics. In solid-state physics [7], it represents a stationary Schrödinger equation of a model of one-dimensional crystal with a more realistic potential than Kronig–Penney or Scarf potentials. This equation, especially its $l = 0$ case, plays an important role in many other fields of physics as well.

For instance, it appeared in some expansions of scattering amplitudes [56], in the study of bifurcations in chaotic Hamiltonian systems [17], it governs distance red-shift for partially filled-beam optics in pressure-free FLRW cosmology [57], it was used in the study of static $SU(2)$ BPS monopoles [9] and kink solutions [58] in the field theory.

4.1. Construction of self-isospectral extension

The spectrum of the one-dimensional periodic system governed by the Hamiltonian operator corresponding to (4.1)

$$H^{-}_{m,l} = -D^2 - C_m \,\mathrm{dn}^2 x - C_l \frac{k'^2}{\mathrm{dn}^2 x} \tag{4.2}$$

consists of the valence bands and the prohibited zones (gaps) which alternate mutually until energy reaches a semi-infinite band of conductance. Configuration of the spectral bands depends sensitively on the constant parameters. As long as m and l acquire integer values, which we suppose to be the case from now on, the spectral bands are arranged such that only finite number of gaps appear. The period $2L$ of the potential with $C_m \neq C_l$ in (4.2) is equal to $2K$, where $K = \int_0^{\pi/2}(1 - k^2 \sin^2\phi)^{-\frac{1}{2}}\,\mathrm{d}\phi$ is the complete elliptic integral of the first kind. The case $C_m = C_l$ corresponds to the Lamé system with the same value of C_m but $C_l = 0$ and the period $2L = K$; it is discussed separately in appendix C. The independent change of parameters $m \to -m-1, l \to -l-1$ leaves the Hamiltonian (4.2) invariant so that we can consider the case $m > l \geqslant 0$ without the loss of generality. In this case the system is m-gap.

To start on the construction of the tri-supersymmetric extension, we focus to the separations of the band-edge states. As we announced, the separation into the periodic and anti-periodic singlets will be used here. Construction of the operators Q_+ and Q_- associated with any separation would require an explicit knowledge of the band-edge states, i.e. an explicit solution of the stationary Schrödinger equation. Fortunately, in the case of natural separation this rather compelling work can be passed with the use of peculiar properties of the model.

The present one-dimensional system is closely related to the finite-dimensional representations of Lie algebra $sl(2,\mathbb{R})$. The Hamiltonian (4.2) can be written as a second-order polynomial in generators of a finite-dimensional irreducible representation of $sl(2,\mathbb{R})$. This important feature underlies quasi-exact solvability of the model, implying that a finite number of eigenstates corresponding to band edges can be found by purely algebraic means [59, 60]. For integer values of m and $l, m > l \geqslant 0$, the space of $2m+1$ singlet states of the associated Lamé system can be treated as a direct sum of two irreducible non-unitary representations of $sl(2,\mathbb{R})$ algebra of dimensions $m-l$ (spin $j_- = \frac{1}{2}(m-l-1)$) and $m+l+1$ $\left(\text{spin } j_+ = \frac{1}{2}(m+l)\right)$ [61, 62].

This fact is deeply related to the structure of the band-edge wavefunctions of the system, $m+l+1$ of which can be factorized formally as

$$\Psi_\mu = \mu \mathcal{F}_\mu(\xi), \tag{4.3}$$

whereas the remaining $m-l$ singlets acquire the form

$$\Psi_\nu = \nu \mathcal{F}_\nu(\xi). \tag{4.4}$$

Here we introduced the functions $\mu = \frac{\mathrm{cn}^{m+l} x}{\mathrm{dn}^l x}$, $\nu = \mathrm{cn}^{m-l-1} x\,\mathrm{dn}^{l+1} x$ and a new variable $\xi(x) = \frac{\mathrm{sn}\, x}{\mathrm{cn}\, x}$, that varies smoothly from $-\infty$ to $+\infty$ in the period interval $(-K, K)$. The functions $\mathcal{F}_\mu(\xi)$ and $\mathcal{F}_\nu(\xi)$ are, in general, polynomials of order $m+l$ and $m-l-1$ in ξ and, as we will see, lie in vector spaces of the irreducible representations of $sl(2,\mathbb{R})$ of the dimensions $m+l+1$ and $m-l$.

As $\xi = \xi(x)$ is periodic, the factors μ and ν dictate periodicity or antiperiodicity of the eigenfunctions; for even $m + l$ the wavefunctions (4.3) are periodic while the functions (4.4) are anti-periodic. For $m > l \geqslant 0$, function μ has one node in the interval $(-K, K)$, while ν can have at most one node there. Being the polynomial in ξ, the function $\mathcal{F}_\mu$ ($\mathcal{F}_\nu$) can acquire at most $m+l$ ($m-l-1$) zeros in this interval. Combine these facts with the general properties of periodic and anti-periodic states of Hill's equation discussed in section 2. The resulting picture shows that starting with periodic ground state and anti-periodic states at the edges of the first gap, the gaps with periodic and anti-periodic states at their edges alter with energy increasing until the $(m-l)$th gap is reached. For higher gaps, all the remaining singlets at the edges are of the same nature as edge states of the $(m-l)$th gap, periodic or anti-periodic with even or odd number of nodes, see also [34].

To reveal the algebraic form of (4.2), we shall recover how the Hamiltonian acts on the 'dynamical' part $\mathcal{F}_\mu$ ($\mathcal{F}_\nu$) of the wavefunctions. Transforming out the function μ and writing the result in the variable ξ , we obtain

$$\begin{aligned} h_\mu = (\mu)^{-1} H_{m,l} \mu &= -(k'^2 (T^+)^2 + (1+k'^2)(T^0)^2 + (T^-)^2 + k^2(l-m)T^0) \\ &= -(1+k'^2\xi^2)(1+\xi^2)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + \xi(2k'^2(m+l-1)\xi^2 + 2(m+lk'^2-1) + k^2)\frac{\mathrm{d}}{\mathrm{d}\xi} \\ &\quad - k'^2(m+l)(m+l-1)\xi^2 + \text{const.} \end{aligned} \tag{4.5}$$

Here the operators,

$$T^+ = \xi^2\partial_\xi - (m+l)\xi, \qquad T^0 = \xi\partial_\xi - \frac{m+l}{2}, \qquad T^- = \partial_\xi, \tag{4.6}$$

$$[T^+, T^-] = -2T^0, \qquad [T^0, T^\pm] = \pm T^\pm, \tag{4.7}$$

are the generators of irreducible representation of $sl(2,\mathbb{R})$ acting on the vector space spanned by monoms $\{1, \xi, \ldots, \xi^{m+l}\}$. Representation (4.6) is specified by the eigenvalue $j_+(j_+ + 1)$ of the Casimir $C = -(T^0)^2 + \frac{1}{2}(T^+T^- + T^-T^+)$ corresponding to $sl(2,\mathbb{R})$ spin $j_+ = \frac{1}{2}(m+l)$.

The other algebraic form of (4.2), h_ν, that acts on the 'dynamical' part $\mathcal{F}_\nu$ of the wavefunctions (4.4), can be obtained by performing the gauge transformation with the other common factor ν. Alternatively, we can use the apparent symmetry $\mu|_{l\to -l-1} = \nu$ and write h_ν immediately just by substituting $l \to -l-1$ into (4.5) and (4.7),

$$h_\nu = h_\mu|_{l\to -l-1} = (\nu)^{-1} H^-_{m,l} \nu = -(k'^2(\tilde{T}^+)^2 + (1+k'^2)(\tilde{T}^0)^2 + (\tilde{T}^-)^2 + k^2(-l-1-m)\tilde{T}^0). \tag{4.8}$$

We denoted by $\tilde{T}^\rho = T^\rho|_{l\to -l-1}$, $\rho = 0, +, -$, the $sl(2,\mathbb{R})$ generators of $(m-l)$-dimensional representation, where T^ρ are the generators (4.6) of spin-j_+ representation. Note that the 'effective' Hamiltonian h_μ is Hermitian with respect to a scalar product defined with a nontrivial weight, $(f, g) = \int_{-\infty}^{\infty} f^*(\xi) g(\xi)(1+k'^2\xi^2)^{-l+\frac{1}{2}}(1+\xi^2)^{-m+\frac{1}{2}}\,\mathrm{d}\xi$; the same is true for h_ν with the change $l \to -l-1$.

Now the background of the natural separation of the singlet states is clear. The periodic and antiperiodic singlet states carry two different irreducible representations of $sl(2,\mathbb{R})$ of dimensions $m+l+1$ and $m-l$. The number of the periodic and anti-periodic singlet states depends on the values of m and l, while their total number $2m+1$ is fixed by the number of gaps m. For instance, for $m = 3, l = 0$ there are $m - l = 3$ periodic band-edge states while for $m = 3, l = 1$ we have $m + l + 1 = 5$ periodic singlet states. To avoid possible confusions during the construction of the supercharges, let us change the notation slightly. We will denote by $X^-_{m,l}$ an operator which annihilates all the functions (4.4), and the operator annihilating all the states (4.3) will be $Y^-_{m,l}$.

First, let us consider eigenstates (4.3) covered by the $(m+l+1)$-dimensional representation of $sl(2,\mathbb{R})$. An operator of the order $m+l+1$ which annihilates the representation space spanned by the monoms $\{1,\xi,\ldots,\xi^{m+l}\}$ has the following general form:

$$y^-_{m,l}=\alpha_{m,l}\partial_\xi^{m+l+1}. \tag{4.9}$$

The function $\alpha_{m,l}$ is fixed uniquely as we require the coefficient at D^{m+l+1} of the operator $Y^-_{m,l}=\mu y^-_{m,l}\frac{1}{\mu}|_{\xi=\xi(x)}$ to be equal to 1. It reads explicitly $\alpha_{m,l}=\left(\frac{\mathrm{dn}\,x}{\mathrm{cn}^2x}\right)^{m+l+1}$. We present below two equivalent forms of the operator $Y^-_{m,l}$. The second, factorized expression, will be particularly helpful in study of the limit case $k\to 1$. An explanation of how to get it from (4.9) can be found in [63],

$$\begin{aligned}Y^-_{m,l}&=D^{m+l+1}+\sum_{j=1}^{m+l+1}c_j^Y D^{m+l+1-j}=\frac{\mathrm{dn}^{m+1}x}{\mathrm{cn}^{m+l+2}x}\left(\frac{\mathrm{cn}^2x}{\mathrm{dn}\,x}D\right)^{m+l+1}\frac{\mathrm{dn}^l x}{\mathrm{cn}^{m+l}x}\\&=\prod_{j=-(m+l)/2}^{(m+l)/2}\left(D+\left(\frac{k^2(m-l)\,\mathrm{cn}^2x}{2}-j\,(k'^2+\mathrm{dn}^2x)\right)\frac{\mathrm{sn}\,x}{\mathrm{cn}\,x\,\mathrm{dn}\,x}\right).\end{aligned} \tag{4.10}$$

The upper index of the ordered product corresponds to the first term on the left-hand side while the lower index denotes the last term on the right-hand side of the product. We can construct the operator $X^-_{m,l}$ in the same way or just by making the substitution $l\to -l-1$ in (4.10) which interchanges considered algebraic schemes. Explicitly, we get

$$\begin{aligned}X^-_{m,l}&=D^{m-l}+\sum_{j=1}^{m-l}c_j^X D^{m-l-j}=\frac{\mathrm{dn}^{m+1}x}{\mathrm{cn}^{m-l+1}x}\left(\frac{\mathrm{cn}^2x}{\mathrm{dn}\,x}D\right)^{m-l}\frac{\mathrm{dn}^{-l-1}x}{\mathrm{cn}^{m-l-1}x}\\&=\prod_{j=-(m-l-1)/2}^{(m-l-1)/2}\left(D+\left(\frac{k^2(m+l+1)\,\mathrm{cn}^2x}{2}-j\,(k'^2+\mathrm{dn}^2x)\right)\frac{\mathrm{sn}\,x}{\mathrm{cn}\,x\,\mathrm{dn}\,x}\right).\end{aligned} \tag{4.11}$$

As we explained in the preceding section, the coefficients of the second highest derivative of $X^-_{m,l}$ and $Y^-_{m,l}$ coincide and enter the explicit construction of the superpartner Hamiltonian,

$$c_1\equiv c_1^X=c_1^Y=-\frac{W'_{m,l}}{W_{m,l}}=k^2\frac{(C_m-C_l)}{2}\frac{\mathrm{sn}\,x\,\mathrm{cn}\,x}{\mathrm{dn}\,x}=\frac{1}{2}(m-l)(m+l+1)k^2\frac{\mathrm{cn}\,x\,\mathrm{sn}\,x}{\mathrm{dn}\,x}. \tag{4.12}$$

The equality $c_1^X=c_1^Y$ reflects the coincidence of the Wronskians of the kernels of $X^-_{m,l}$ and $Y^-_{m,l}$ up to inessential numerical factor related to the arbitrariness in normalization of their zero modes. The essential part of these Wronskians is given by the nodeless function

$$W_{m,l}(x)=(\mathrm{dn}\,x)^{\frac{1}{2}(m-l)(m+l+1)}, \tag{4.13}$$

whose invariance with respect to the change $l\to -l-1$ just reflects the indicated equality of the coefficients.

The Jacobi function property $\mathrm{dn}(x+K)=k'/\mathrm{dn}\,x$ shows that the Wronskian $W_{m,l}(x)$ satisfies the relation (3.36), and supports our conjecture that the natural separation of the singlet states into periodic and anti-periodic states results in the self-isospectral tri-supersymmetric system with the partner Hamiltonian operator $\tilde{H}\equiv H^+_{m,l}$ to be the original Hamiltonian translated for the half-period, $H^+_{m,l}(x)=H^-_{m,l}(x+K)$. Its explicit form is

$$H^+_{m,l}=H^-_{m,l}+2c_1'=-D^2-C_l\,\mathrm{dn}^2x-C_m\frac{k'^2}{\mathrm{dn}^2x}. \tag{4.14}$$

Recalling (3.17), the generator of the hidden bosonized supersymmetry of the associated Lamé system (4.2) acquires the following factorized form:

$$Z^-_{m,l} = \frac{\mathrm{dn}^{-l}x}{\mathrm{cn}^{m-l+1}x}\left(\frac{\mathrm{cn}^2x}{\mathrm{dn}\,x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m-l}\left(\frac{\mathrm{dn}\,x}{\mathrm{cn}\,x}\right)^{2m+1}\left(\frac{\mathrm{cn}^2x}{\mathrm{dn}\,x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+l+1}\frac{\mathrm{dn}^l x}{\mathrm{cn}^{m+l}x} \tag{4.15}$$

$$= \frac{\mathrm{dn}^{-l}x}{\mathrm{sn}^{m-l+1}x}\left(\frac{\mathrm{sn}^2x}{\mathrm{dn}\,x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m-l}\left(\frac{\mathrm{dn}\,x}{\mathrm{sn}\,x}\right)^{2m+1}\left(\frac{\mathrm{sn}^2x}{\mathrm{dn}\,x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^{m+l+1}\frac{\mathrm{dn}^l x}{\mathrm{sn}^{m+l}x}, \tag{4.16}$$

where we used alternative expressions $X^-_{m,l}(x) = \frac{\mathrm{dn}^{m+1}x}{\mathrm{sn}^{m-l+1}x}\left(\frac{\mathrm{sn}^2x}{\mathrm{dn}\,x}D\right)^{m-l}\frac{\mathrm{dn}^{-l-1}x}{\mathrm{sn}^{m-l-1}x}$ and $Y^-_{m,l}(x) = \frac{\mathrm{dn}^{m+1}x}{\mathrm{sn}^{m+l+2}x}\left(\frac{\mathrm{sn}^2x}{\mathrm{dn}\,x}D\right)^{m+l+1}\frac{\mathrm{dn}^l x}{\mathrm{sn}^{m+l}x}$, obtained with the use of a specific identity

$$\frac{1}{\mathrm{sn}^{j+1}x}\left(\frac{\mathrm{sn}^2x}{\mathrm{dn}\,x}D\right)^j\frac{1}{\mathrm{sn}^{j-1}x} = \frac{1}{\mathrm{cn}^{j+1}x}\left(\frac{\mathrm{cn}^2x}{\mathrm{dn}\,x}D\right)^j\frac{1}{\mathrm{cn}^{j-1}x}. \tag{4.17}$$

By making use of the same identity we can prove that $Y^-_{m,l}(x+K) = (-1)^{m+l+1}\big(Y^-_{m,l}(x)\big)^\dagger$ and $X^-_{m,l}(x+K) = (-1)^{m-l}\big(X^-_{m,l}(x)\big)^\dagger$. Finally, we write the obtained extended Hamiltonian $\mathcal{H}$ as well as the Hermitian diagonal and antidiagonal integrals

$$\mathcal{H}_{m,l} = \begin{pmatrix} H^+_{m,l} & 0 \\ 0 & H^-_{m,l}\end{pmatrix} = \begin{pmatrix} H^-_{m,l}(x+K) & 0 \\ 0 & H^-_{m,l}(x)\end{pmatrix}, \tag{4.18}$$

$$\mathcal{Z}_{m,l} = \mathrm{i}^{2m+1}\begin{pmatrix} Z^+_{m,l} & 0 \\ 0 & Z^-_{m,l}\end{pmatrix} = \mathrm{i}^{2m+1}\begin{pmatrix} Z^-_{m,l}(x+K) & 0 \\ 0 & Z^-_{m,l}(x)\end{pmatrix}, \tag{4.19}$$

$$\mathcal{X}_{m,l} = \mathrm{i}^{m-l}\begin{pmatrix} 0 & X^-_{m,l} \\ X^+_{m,l} & 0\end{pmatrix} = \mathrm{i}^{m-l}\begin{pmatrix} 0 & X^-_{m,l}(x) \\ X^-_{m,l}(x+K) & 0\end{pmatrix}, \tag{4.20}$$

$$\mathcal{Y}_{m,l} = \mathrm{i}^{m+l+1}\begin{pmatrix} 0 & Y^-_{m,l} \\ Y^+_{m,l} & 0\end{pmatrix} = \mathrm{i}^{m+l+1}\begin{pmatrix} 0 & Y^-_{m,l}(x) \\ Y^-_{m,l}(x+K) & 0\end{pmatrix}, \tag{4.21}$$

which represent an explicit realization of (3.19). In accordance with the analysis of section 3, the integrals $\mathcal{Q}_\pm$ are identified with $\mathcal{X}_{m,l}$ and $\mathcal{Y}_{m,l}$ in the following way: $\mathcal{Q}_- = \mathcal{X}_{m,l}$ and $\mathcal{Q}_+ = \mathcal{Y}_{m,l}$ when $m-l$ is odd, and their roles are interchanged when $m-l$ is even.

4.2. Self-isospectral pairs and 'superpotential'

Let us indicate on an interesting representation of the self-isospectral pairs of the Hamiltonians that generalizes a representation $H_\pm = -D^2 + W^2 \pm W'$ of the superpartner Hamiltonians in terms of the superpotential in the case of the usual (linear) $N = 2$ supersymmetry.

The self-isospectral pair (4.2), (4.14) can be presented in the equivalent form

$$H^\pm_{m,l} = -D^2 + 2\frac{C_m + C_l}{(C_m - C_l)^2}(\ln W_{m,l})'^2 \pm (\ln W_{m,l})'' + (1+k'^2)\frac{1}{2}(C_m + C_l), \tag{4.22}$$

where $W_{m,l}$ is the Wronskian (4.13) corresponding to the kernels of operators $X^-_{m.l}$ and $Y^-_{m,l}$. Let us denote $C_+ = \sqrt{\frac{1}{2}(C_m + C_l)}$, $C_- = \frac{1}{2}(m-l)(m+l+1) = \frac{1}{2}(C_m - C_l)$, and define a function

$$\mathcal{W} = -(\ln \mathrm{dn}^{C_+}x)'. \tag{4.23}$$

Then (4.22) can be rewritten equivalently as

$$H^\pm_{m,l} - (1+k'^2)C_+^2 = -D^2 + \mathcal{W}^2 \pm \frac{C_-}{C_+}\mathcal{W}'. \tag{4.24}$$

Equation (4.24) is reminiscent of supersymmetric quantum mechanics representation (3.7). This is not just a coincidence. In the case $m - l = 1$, the system $H^{-}_{m,m-1}$ is characterized by the presence of only one periodic singlet $\Psi_0 = \mathrm{dn}^m x$, which is the ground state with energy corresponding to a subtracted constant term on the left-hand side of equation (4.24). In this case the first-order supercharge $\mathcal{X}_{m,m-1}$ reduces to one of the first-order supercharges (3.8) of $N = 2$ supersymmetry, (4.23) takes a form of a usual representation of a superpotential in terms of the ground state, and (4.24) reduces to (3.7).

There exists a simple generalization of a classical model for supersymmetric quantum mechanics to the case of nonlinear supersymmetry of order $n > 1$ [32]. It consists in the change of the boson–fermion coupling term $\theta^+\theta^- W'$ in classical Hamiltonian for $n\theta^+\theta^- W'$, where $\theta^+\theta^-$ is a classical analog for σ_3 and $\theta^{\pm}$ are Grassmann variables describing fermion degrees of freedom. However, unlike the linear case $n = 1$, for $n > 1$ such a generalization suffers a problem of the quantum anomaly, which can be solved in a general form only for $n = 2$ [33]. One can show that for $m - l = 2$, when the supercharge $\mathcal{X}_{m,m-2}$ is the differential operator of the second order, representation (4.24) is in correspondence with the solution of the quantum anomaly problem for $n = 2$.

Let us stress that for $m - l > 1$ the argument of logarithm in the definition of the superpotential-like function (4.23) does not correspond to a ground state of the system[8]. It is just the appropriately rescaled logarithmic derivative of the Wronskian (4.13), $\mathcal{W} = -\frac{C_+}{C_-}(\ln W_{m,l})'$.

4.3. Some examples

Due to the general result presented in section 3, there exist $2^m - 1$ isospectral extensions of an m-gap associated Lamé Hamiltonian. The new systems can be obtained from the initial system by sequent use of Darboux transformation and the transformation of Crum of the second order. Moreover, the complete set of 2^m isospectral systems can be sorted out into 2^{m-1} self-isospectral tri-supersymmetric pairs. We present here an example of Lamé system to illustrate this picture explicitly.

First, we explain briefly some subtleties related to the sequent use of the transformation of Crum. Let us consider a two-gap associated Lamé system represented by Hamiltonian H. There are three admissible separations of the singlet states: sorting out the states at the edges of the first gap, or of the second gap, or of both of them. Let us denote the supercharges associated with these separations as $Q_{\pm,j}$, $j = 1, 2, 3$. Then we can construct three isospectral superpartner Hamiltonians $H_{(j)}$ satisfying

$$H_{(j)} Q_{\pm,j} = Q_{\pm,j} H. \tag{4.25}$$

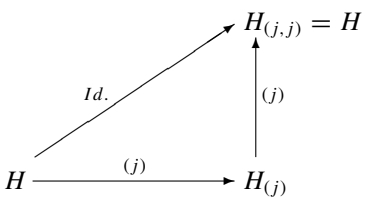
We can repeat the procedure with anyone of the new systems, obtaining another isospectral Hamiltonian $H_{(k,j)}$ (j refers to the system $H_{(j)}$, k denotes the next choice of separation). Although we can make this procedure repeatedly, it will generate only limited number of new systems. This is due to simple rules following the sequent use of transformations of Crum.

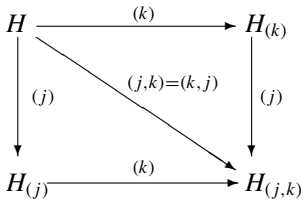
The first rule: when we choose sequently two identical separations, we return to the initial system. In our two-gap setting, let us start with the first separation and construct the new Hamiltonian H_1 with the help of operator $Q_{+,1}$. The operator which annihilates the states at the edges of the first gap of H_1 coincides with $Q^{\dagger}_{+,1}$. So, repeating the procedure with H_1 with the same separation, we obtain a new Hamiltonian $H_{(1,1)}$ which is related to initial H by the intertwining relation $H_{(1,1)} Q^{\dagger}_{\pm,1} Q_{\pm,1} = Q^{\dagger}_{\pm,1} Q_{\pm,1} H$. But the operator $Q^{\dagger}_{\pm,1} Q_{\pm,1}$ is a polynomial in H, and we get $H_{(1,1)} = H$. In general, there holds $H_{(j,j)} = H$.

[8] For the explicit form of the ground states of the associated Lamé system with some values of m and l see [62].

The other rule tells that the choice of sequent separations is 'commutative'. Construct the Hamiltonian H_1 using the supercharge $Q_{+,1}$. Then, choosing the second separation, we construct $H_{(2,1)}$ with help of operator $\tilde{Q}_{+,2}$ which annihilates the states at the edges of the second gap of the system H_1. Hamiltonian $H_{(2,1)}$ satisfies a relation $H_{(2,1)}\tilde{Q}_{\pm,2}Q_{\pm,1} = \tilde{Q}_{\pm,2}Q_{\pm,1}H$. The operator $Q_{+,3} = \tilde{Q}_{\pm,2}Q_{\pm,1}$ is of the fourth order and can be factorized in different ways. For instance, its alternative factorization is $Q_{+,3} = \tilde{Q}_{\pm,1}Q_{\pm,2}$, that corresponds to the interchanged choices of the separations. Speaking in general, there holds $H_{(k,j)} = H_{(j,k)}$.

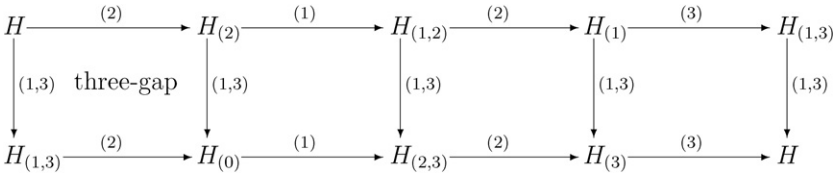
Let us denote shortly (j) (or (j,k)) a transformation of Crum of the second (or of the fourth) order which annihilates singlet states at the edges of the jth gap (or of the both jth and kth gaps). In this notation, Darboux transformation associated with the ground state is represented by (0). Coherently, we denote $H_{(j)}$ or $H_{(j,k)}$ the Hamiltonians obtained by these transformations. Then the rules for sequent use of Darboux–Crum transformations can be depicted by the following schemes





where $Id.$ represents an identity operator.

In the two-gap case, we can find three new isospectral Hamiltonians in this way. Let us present isospectral transformations of the $(m = 3, l = 0)$ Lamé Hamiltonian with their relation to the original system. Seven new isospectral Hamiltonians are found and four self-isospectral pairs can be formed. In the following scheme, any of the new systems can be reached by sequent application of Darboux (0) or second-order Crum's transformation (k) on the initial Hamiltonian H. The vertical lines correspond to transformation representing the natural separation so that the Hamiltonians related by these lines form self-isospectral pairs.



As it can be observed in figure 2, the nature of the potentials of the obtained systems is distinct from the original ones. However, the spectrum of the corresponding Hamiltonians is completely identical. The potentials can be tuned with modular parameter k that broadens the applicability of these systems. The infinite-period limit of the self-isospectral extension of the associated Lamé systems is discussed in detail in the forthcoming section.

5. Superextended Pöschl–Teller system and infinite-period limit of tri-supersymmetry

In this section, we analyze in detail the infinite-period limit of the tri-supersymmetric self-isospectral extension (4.18)–(4.21) of the associated Lamé system. We explain how the structure of tri-supersymmetry is modified and study its implications, in particular the restoration of the unbroken tri-supersymmetry.

Infinite-period limit corresponds to $k \to 1$ $(k' \to 0)$. In this limit, the associated Lamé Hamiltonian changes to the energy operator of the Pöschl–Teller system. The band structure transforms in the following way. The states of the conduction band are transformed into the

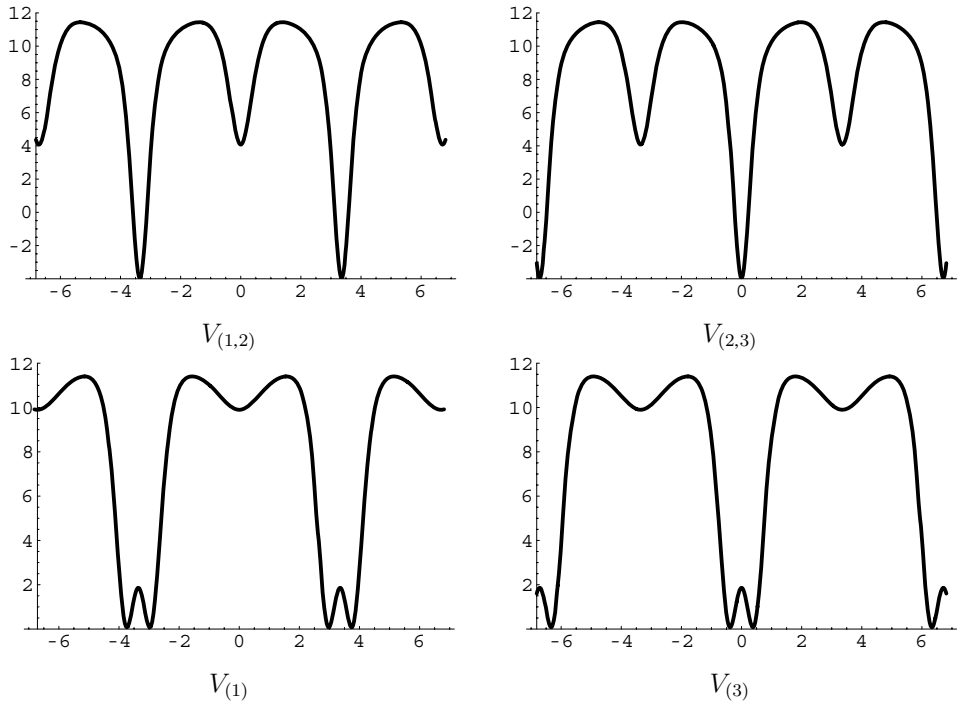


Figure 2. The Hamiltonians $H_{(j,k)} = -D^2 + V_{(j,k)} - 12$ with the plotted potentials are spectrally identical (see the last diagram in section 4). The potentials on the left and right are mutually shifted in half of the period. The modular parameter is set $k = 0.99$.

states of the scattering sector of the spectrum, and the singlet edge state of the conduction band is transformed into the first (lowest) singlet state of the continuous spectrum. The valence bands shrink, two band-edge states corresponding to the same permitted band converge smoothly in a unique wavefunction. This wavefunction can be non-physical in one of the limit systems.

5.1. Self-isospectral supersymmetry in the infinite-period limit

When k tends to 1, the Jacobi elliptic functions cease to be doubly periodic as their real period extends to infinity while the complex period takes a finite value $2\mathrm{i}K' = 2\mathrm{i}\pi$; they transform into the hyperbolic functions, $\mathrm{dn}\,x \to \mathrm{sech}\,x$, $\mathrm{cn}\,x \to \mathrm{sech}\,x$, $\mathrm{sn}\,x \to \tanh x$. In this limit, the superextended setting described by the two mutually shifted periodic Hamiltonians (4.18) acquires the following form:

$$\mathcal{H}_{m,l} = \begin{pmatrix} H^+_{m,l} & 0 \\ 0 & H^-_{m,l} \end{pmatrix} \underset{k=1}{\longrightarrow} \mathcal{H}^{PT}_{m,l} = \begin{pmatrix} \hat{H}^+_l & 0 \\ 0 & \hat{H}^-_m \end{pmatrix}, \tag{5.1}$$

where the resulting operators represent two systems with the Pöschl–Teller potential of different interaction strengths specified by the integers m and l,

$$\hat{H}^-_m = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - C_m\,\mathrm{sech}^2 x, \qquad \hat{H}^+_l = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - C_l\,\mathrm{sech}^2 x. \tag{5.2}$$

The system keeps the parity invariance, $\left[R, \mathcal{H}^{PT}_{m,l}\right] = 0$. As we deal with integer values of m and l, Hamiltonians (5.2) are reflectionless, i.e. the transmission coefficients are equal to 1.

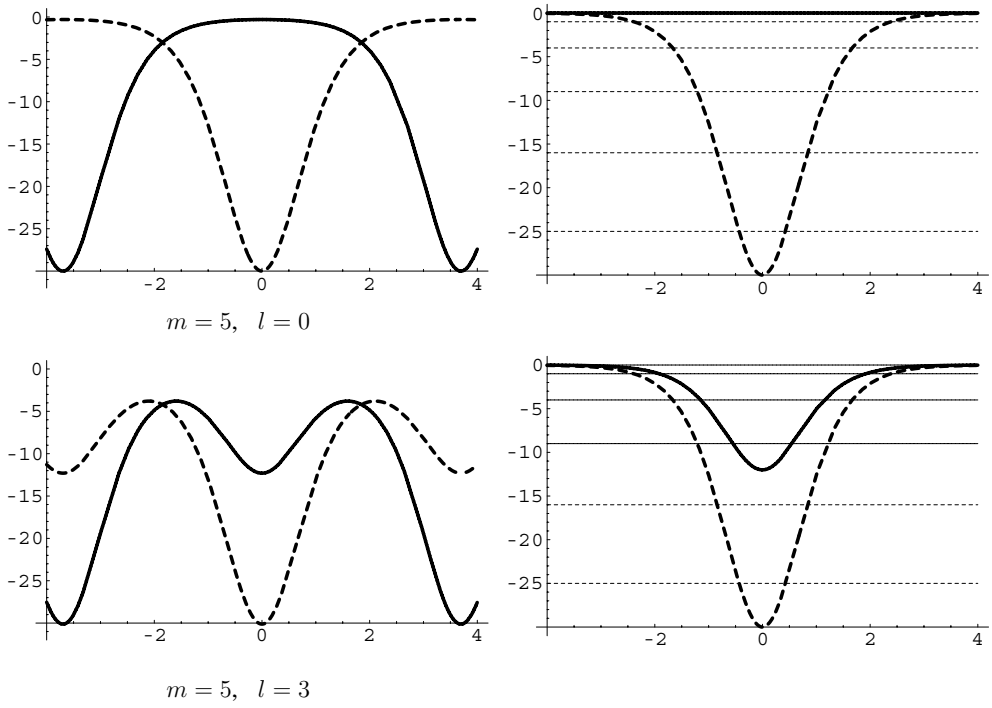


Figure 3. Potentials on the left correspond to $H^-_{m,l}$ (dashed thick line) and $H^+_{m,l}$ (solid thick line); $k^2 = 0.99$. On the right, the limit $k \to 1$ of these potential functions is shown. The solid thin lines represent the shared bound states and the lowest scattering state. Dashed thin lines represent $m - l = 2$ singlet states.

The Hamiltonian $\hat{H}^-_m \left(\hat{H}^+_l\right)$ possesses $m + 1$ $(l + 1)$ singlet states, m (l) of them are bound states, the remaining one corresponds to the lowest state in the scattering sector. Hamiltonians (5.2) are almost isospectral. Their spectra coincide in the continuous part $E \in [0, \infty)$ and just in $l + 1$ singlet states. The Hamiltonian $\hat{H}^-_m$ has additional $m - l$ lower energy levels. This indicates what happens with the spectral structure of the extended Hamiltonian $\mathcal{H}_{m,l}$ when we stretch the real period into the infinity. The $2m + 1$ band-edge states of $H^-_{m,l}$ transform into $m + 1$ physical states of $\hat{H}^-_m$, while in the case of the superpartner system $H^+_{m,l}$ only $2l + 1$ band-edge states of highest excitations converge to the physical wavefunctions, the rest is physically unacceptable. Thus the doublets of band-edge states and quadruplets of the quasi-periodic states change into $m - l$ singlets and $l + 1$ doublet states, and into the quadruplets of the scattering states (see figure 3). The presence of the singlet states gives a taste of a different nature of the tri-supersymmetry which we discuss in what follows.

The Hamiltonians (5.2) admit the following representation:

$$\hat{H}^-_m = -\mathcal{D}_{-m}\mathcal{D}_m - m^2, \qquad \hat{H}^+_l = -\mathcal{D}_{-l}\mathcal{D}_l - l^2, \tag{5.3}$$

where the definition and the basic properties of the operator $\mathcal{D}_n$ are

$$\mathcal{D}_n = D + n \tanh x, \qquad \mathcal{D}^\dagger_n = -\mathcal{D}_{-n}, \qquad \mathcal{D}_{-n}\mathcal{D}_n = \mathcal{D}_{n+1}\mathcal{D}_{-n-1} + (2n + 1). \tag{5.4}$$

We shall focus to the properties of tri-supersymmetry and of the supercharges (4.19)–(4.21) in particular. Taking the limit $k = 1$, the non-diagonal integrals (4.20) and (4.21) transform

as follows:

$$\mathcal{X}_{m,l} = \mathrm{i}^{m-l}\begin{pmatrix} 0 & X^-_{m,l} \\ X^+_{m,l} & 0 \end{pmatrix} \underset{k=1}{\longrightarrow} \mathcal{X}^{PT}_{m,l} = \mathrm{i}^{m-l}\begin{pmatrix} 0 & \hat{X}^-_{m,l} \\ \hat{X}^+_{m,l} & 0 \end{pmatrix}, \tag{5.5}$$

$$\mathcal{Y}_{m,l} = \mathrm{i}^{m+l+1}\begin{pmatrix} 0 & Y^-_{m,l} \\ Y^+_{m,l} & 0 \end{pmatrix} \underset{k=1}{\longrightarrow} \mathcal{Y}^{PT}_{m,l} = \mathrm{i}^{m+l+1}\begin{pmatrix} 0 & \hat{Y}^-_{m,l} \\ \hat{Y}^+_{m,l} & 0 \end{pmatrix}, \tag{5.6}$$

where the non-diagonal components are

$$\hat{Y}^-_{m,l} = \mathcal{D}_{-l}\mathcal{D}_{-l+1}\cdots\mathcal{D}_{m-1}\mathcal{D}_m, \qquad \hat{Y}^+_{m,l} = \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_{l-1}\mathcal{D}_l, \tag{5.7}$$

$$\hat{X}^-_{m,l} = \mathcal{D}_{l+1}\mathcal{D}_{l+2}\cdots\mathcal{D}_{m-1}\mathcal{D}_m, \qquad \hat{X}^+_{m,l} = \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_{-l-2}\mathcal{D}_{-l-1}, \tag{5.8}$$

and $\hat{X}^+_{m,l} = (-1)^{m-l}\big(\hat{X}^-_{m,l}\big)^\dagger$, $\hat{Y}^+_{m,l} = (-1)^{m+l+1}\big(\hat{Y}^-_{m,l}\big)^\dagger$. The limit does not violate commutation relations (tri-supersymmetry is maintained) and we have

$$\big[\mathcal{H}^{PT}_{m,l}, \mathcal{X}^{PT}_{m,l}\big] = \big[\mathcal{H}^{PT}_{m,l}, \mathcal{Y}^{PT}_{m,l}\big] = \big[\mathcal{X}^{PT}_{m,l}, \mathcal{Y}^{PT}_{m,l}\big] = 0. \tag{5.9}$$

The components of the squares of the non-diagonal supercharges,

$$\big(\mathcal{X}^{PT}_{m,l}\big)^2 = \begin{pmatrix} \hat{X}^-_{m,l}\hat{X}^+_{m,l} & 0 \\ 0 & \hat{X}^+_{m,l}\hat{X}^-_{m,l} \end{pmatrix}, \qquad \big(\mathcal{Y}^{PT}_{m,l}\big)^2 = \begin{pmatrix} \hat{Y}^-_{m,l}\hat{Y}^+_{m,l} & 0 \\ 0 & \hat{Y}^+_{m,l}\hat{Y}^-_{m,l} \end{pmatrix}, \tag{5.10}$$

correspond to the integrals of motion of individual Pöschl–Teller subsystems. The results of the previous section suggest that these will be proportional to certain spectral-type polynomials in Hamiltonian. This is the case indeed. However, the situation changes significantly comparing with the periodic system. With the sequent use of the identity in (5.4), we can derive

$$\big(\mathcal{X}^{PT}_{m,l}\big)^2 = \prod_{j=0}^{m-l-1}\big(\mathcal{H}^{PT}_{m,l} - E_{m,j}\big) = P^{PT}_X\big(\mathcal{H}^{PT}_{m,l}\big). \tag{5.11}$$

Here $E_{m,j} = -(m-j)^2$, $j = 0, \ldots, m-l-1$, correspond to the $m-l$ singlet states of the superextended system ($m-l$ bound states correspond to the lowest energies of $\hat{H}^-_m$ or, equivalently to the $m-l$ nonphysical states of $\hat{H}^+_l$). The square of the second non-diagonal supercharge can be factorized with use of P^{PT}_X

$$\big(\mathcal{Y}^{PT}_{m,l}\big)^2 = \big(\mathcal{H}^{PT}_{m,l} - E_{m,m}\big)\prod_{j=m-l}^{m-1}\big(\mathcal{H}^{PT}_{m,l} - E_{m,j}\big)^2 P^{PT}_X\big(\mathcal{H}^{PT}_{m,l}\big), \tag{5.12}$$

Apart from the roots shared with P^{PT}_X, there also appear the $l+1$ doubly-degenerate energies of the superextended system (5.1). The l bound-state energies $E_{m,j} = -(m-j)^2$, $j = m-l, \ldots, m-1$, are the double roots, and the energy of the lowest state of the continuous spectrum, $E_{m,m} = 0$, is a simple root.

Considering the limit case of the diagonal supercharge (4.19), we encounter an interesting situation. In [64] it was observed that for a *reflectionless* Pöschl–Teller (PT) system, there exists a hidden bosonized supersymmetry. If the system has n bound states (and hence $n+1$ singlet states), there is a parity-odd integral of motion of order $2n+1$,

$$\mathcal{A}_{2n+1} = \mathcal{D}_{-n}\mathcal{D}_{-n+1}\cdots\mathcal{D}_0\cdots\mathcal{D}_{n-1}\mathcal{D}_n, \tag{5.13}$$

which annihilates all the singlet states and some *non-physical* states, whose origin was clarified in [65] from the point of view of the Lamé equation and its hidden bosonized supersymmetry[9].

[9] Earlier, higher-order differential operators of this nature were discussed in the context of supersymmetric quantum mechanics in [48, 66, 67]. However, their sense and the intimate relation with the algebro-geometric potentials were not understood, see also footnote 7.

The subsystems $\hat{H}^-_m$ and $\hat{H}^+_l$ have odd-integrals of motion $\mathcal{A}_{2m+1}$ and $\mathcal{A}_{2l+1}$ of orders $2m+1$ and $2l+1$, respectively. On the other hand, we know that the diagonal components of integral $\mathcal{Z}^{PT}$ are the parity-odd integrals of order $2m+1$ for each subsystem. Particularly, $\hat{H}^+_l$ would have two parity-odd symmetries of different orders. Let us explain the picture and show how the integrals $\mathcal{A}_{2m+1}$ an $\mathcal{A}_{2l+1}$ manifest their presence in the tri-supersymmetric scheme.

In the limit $\mathcal{Z}_{m,l} \underset{k=1}{\longrightarrow} \mathcal{Z}^{PT}_{m,l}$, we can trace the presence of the integrals $\mathcal{A}_{2m+1}$ and $\mathcal{A}_{2l+1}$ in the diagonal components

$$Z^-_{m,l} \underset{k=1}{\longrightarrow} \hat{Z}^-_{m,l} = \hat{X}^+_{m,l}\hat{Y}^-_{m,l} = \mathcal{A}_{2m+1} = \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_0\cdots\mathcal{D}_{m-1}\mathcal{D}_m, \tag{5.14}$$

$$Z^+_{m,l} \underset{k=1}{\longrightarrow} \hat{Z}^+_{m,l} = \hat{X}^-_{m,l}\hat{Y}^+_{m,l} = \hat{X}^-_{m,l}\hat{X}^+_{m,l}\mathcal{A}_{2l+1}. \tag{5.15}$$

Each of these operators (it is worth noting again that they have the same order) is the integral for the corresponding subsystem, and together they annihilate the singlet and doublet states of the super-extended system. $\left(\mathcal{Z}^{PT}_{m,l}\right)^2$ produces a polynomial of the form

$$\left(\mathcal{Z}^{PT}_{m,l}\right)^2 = \left(\mathcal{H}^{PT}_{m,l} - E_{m,m}\right)\prod_{j=0}^{m-1}\left(\mathcal{H}^{PT}_{m,l} - E_{m,j}\right)^2, \tag{5.16}$$

which can be related to a *degenerate* spectral hyperelliptic curve of genus m, in contrary to the m-gap system (4.18), whose spectral polynomial (3.22) corresponds to a non-degenerate hyperelliptic curve of the same genus. It reflects the fact that the band structure disappeared; every two band-edge states of the same band transform into a single bound state which ends up in the degeneracy in the spectral polynomial. Accordingly, the degeneracy does not appear for the lowest state of the continuous spectrum.

The components of the integral $\mathcal{Y}^{PT}_{m,l}$ can be rewritten in the following way:

$$\hat{Y}^-_{m,l} = \mathcal{A}_{2l+1}\mathcal{D}_{l+1}\cdots\mathcal{D}_{m-1}\mathcal{D}_m = \mathcal{A}_{2l+1}\hat{X}^-_m, \qquad \hat{Y}^+_{m,l} = \hat{X}^+_l\mathcal{A}_{2l+1}. \tag{5.17}$$

The relation (5.12) can be expressed also as

$$\left(\mathcal{Y}^{PT}_{m,l}\right)^2 = \mathcal{A}^2_{2l+1}\left(\mathcal{H}^{PT}_{m,l}\right)P^{PT}_X\left(\mathcal{H}^{PT}_{m,l}\right), \tag{5.18}$$

where in correspondence with (5.13) $\mathcal{A}_{2l+1} = \mathcal{D}_{-l}\mathcal{D}_{-l+1}\cdots\mathcal{D}_0\cdots\mathcal{D}_{l-1}\mathcal{D}_l$. These formulae provide an alternative insight into the kernel of the supercharge $\mathcal{Y}^{PT}_{m,l}$ and its commutation relation with $\mathcal{H}^{PT}_{m,l}$. The later one can be derived independently just with the use of the commutation relations $\left[\mathcal{H}^{PT}_{m,l}, \mathcal{X}^{PT}_{m,l}\right] = 0$ and $\left[\hat{H}^+_l, \mathcal{A}_{2l+1}\right] = 0$. In particular, we have

$$\hat{Y}^-_{m,l}\hat{H}^-_m = \mathcal{A}_{2l+1}\hat{X}^-_{m,l}\hat{H}^-_m = \mathcal{A}_{2l+1}\hat{H}^+_l\hat{X}^-_{m,l} = \hat{H}^+_l\mathcal{A}_{2l+1}\hat{X}^-_{m,l} = \hat{H}^+_l\hat{Y}^-_{m,l} \tag{5.19}$$

In the infinite-period limit, the tight relation of the non-diagonal integrals (5.5) and (5.6) is manifested by means of the parity-odd integral $\mathcal{A}_{2l+1}$ of $\hat{H}^+_l$, see equation (5.17), which was not presented in the periodic case. As a consequence, this integral appears also in the structure of the diagonal integral $\hat{Z}^+_{m,l}$, see (5.15).

5.2. Supercharges action and relation of nonphysical with physical solutions

In the case of associated Lamé system, the action of the operators $\mathcal{X}^-_{m,l}$ and $\mathcal{Y}^-_{m,l}$ was quite clear. Each of them annihilated two disjoint subsets of the whole family of $2m+1$ singlet states of the Hamiltonian $H^-_{m,l}$. In the limit case, the situation ceases to be so transparent. The systems described by (5.2) differ in the number of the singlet states, the Hamiltonian $\hat{H}^-_m$ $\left(\hat{H}^+_l\right)$ has $m+1$ $(l+1)$ bound states. However, the order of the supercharges is not affected by

the limit. There arises a natural question: what kind of functions is annihilated additionally by the nontrivial integrals. We clarify here this intricate situation.

We introduced the operator $\mathcal{D}_n$ (see (5.4)) which proved to be useful in factorization of both the Hamiltonians (5.3) and the supercharges (5.5) and (5.6). We can interpret this operator as a Darboux transformation which satisfies

$$\mathcal{D}_m \hat{H}_m^- = \hat{H}_{m-1}^- \mathcal{D}_m, \tag{5.20}$$

where the new Hamiltonian $\hat{H}_{m-1}$ has $m-1$ bound states. The operator $\mathcal{D}_m$ annihilates the ground state of $\hat{H}_m^-$ so that the corresponding energy level is missing in the spectrum of $\hat{H}_{m-1}$. The Hamiltonians $\hat{H}_m$ and $\hat{H}_{m-1}$ related by $\mathcal{D}_m$ are of the same nature but with a shifted parameter. This phenomenon, mediated by Darboux transformation, is called a *shape invariance*.

We can apply the transformation of Darboux repeatedly, annihilating the lowest bound state in each step. This procedure induces the following sequence of Hamiltonians:

$$\hat{H}_m^- \to \hat{H}_{m-1}^- \to \hat{H}_{m-2}^- \to \cdots \to \hat{H}_{l+1}^- \to \hat{H}_l^- = \hat{H}_l^+. \tag{5.21}$$

The resulting operator which relates $\hat{H}_m^-$ and $\hat{H}_l^+$ can be interpreted as a Crum–Darboux transformation of order $m-l$. This transformation coincides with that produced by the operator $\hat{X}_{m,l}^-$,

$$\hat{X}_{m,l}^- \hat{H}_m^- = \hat{H}_l^+ \hat{X}_{m,l}^-, \qquad \hat{X}_{m,l}^- = \mathcal{D}_{l+1} \cdots \mathcal{D}_m. \tag{5.22}$$

Therefore, $\hat{X}_{m,l}^-$ annihilates $m-l$ bound states of $\hat{H}_m^-$. In the sense of the superextended Hamiltonian $\mathcal{H}_{m,l}^{PT}$, these bound states are *singlets*. The energy levels corresponding to the states annihilated by $\hat{X}_{m,l}^-$ are absent in the spectrum of $\hat{H}_l^+$ (that makes the systems almost isospectral). Let us denote the corresponding $m-l$ bound-state energies of $\hat{H}_m^-$ as E_m. We denote $l+1$ remaining energies of singlet states of $\hat{H}_m^-$ as E_l. These $l+1$ levels are shared by the singlet states of $\hat{H}_l^+$. So, the energies of singlet states of $\hat{H}_m^-$ are formed by E_m and E_l.

The operator $\hat{X}_{m,l}^+$ annihilates a non-physical eigenstate of $\hat{H}_l^+$ corresponding to the energy E_m $\hat{X}_{m,l}^+ \tilde{\psi}_m = 0$, $\hat{H}_l^+ \tilde{\psi}_m = E_m \tilde{\psi}_m$. Acting with the operator $\hat{X}_{m,l}^+$ on the $l+1$ physical states (l bound states and the lowest state of the continuous spectrum) $\tilde{\psi}_l$ of $\hat{H}_l^+$, $\hat{H}_l^+ \tilde{\psi}_l = E_l \tilde{\psi}_l$, we get the bound states of $\hat{H}_m^-$ corresponding to the energy E_l $\hat{X}_{m,l}^+ \tilde{\psi}_l = \psi_l$, $\hat{H}_m^- \psi_l = E_l \psi_l$. The operator $\hat{Y}_{m,l}^+$ annihilates the physical eigenstates $\tilde{\psi}_l$, $\hat{Y}_{m,l}^+ \tilde{\psi}_l = 0$, and also annihilates the second, non-physical, solution $\tilde{\eta}_m$ of the Hamiltonian $\hat{H}_l^+$ corresponding to the eigenvalue E_m, $\hat{Y}_{m,l}^+ \tilde{\eta}_m = 0$, $\hat{H}_l^+ \tilde{\eta}_m = E_m \tilde{\eta}_m$, $\tilde{\eta}_m = \tilde{\psi}_m \int^x \tilde{\psi}_m^{-2} \, \mathrm{d}x$. The function $\tilde{\eta}_m$ is mapped by $\hat{X}_{m,l}^+$ to a physical state ψ_m of $\hat{H}_m^-$ with energy E_m. The same result is obtained when we act by $\hat{Y}_{m,l}^+$ on the state $\tilde{\psi}_m \in \operatorname{Ker} \hat{X}_{m,l}^+$, $\hat{Y}_{m,l}^+ \tilde{\psi}_m = \psi_m$, $\hat{X}_{m,l}^+ \tilde{\eta}_m \propto \psi_m$, $\hat{H}_m^- \psi_m = E_m \psi_m$.

To get an insight into the kernel of the operator $\hat{Y}_{m,l}^-$, it is convenient to rewrite this operator in the factorized form

$$\hat{Y}_{m,l}^- = \mathcal{D}_{-l} \mathcal{D}_{-l+1} \cdots \mathcal{D}_{-1} \underbrace{\mathcal{D}_0 \mathcal{D}_1 \cdots \mathcal{D}_{m-1} \mathcal{D}_m}_{\text{annihilates singlets}}. \tag{5.23}$$

The indicated right part of the operator annihilates all the $m+1$ singlet states of $\hat{H}_m^-$. Due to the remaining part, there are additional annihilated l functions ϕ_j, $j = m-l, \ldots, m-1$. These additional functions need not to be solutions of the Schrödinger equation corresponding to $\hat{H}_m^-$. Indeed, due to representation

$$\hat{Y}_{m,l}^+ \hat{Y}_{m,l}^- = (\hat{H}_m^- - E_{m,m}) \prod_{j=m-l}^{m-1} (\hat{H}_m^- - E_{m,j})^2 P_X^{PT}(\hat{H}_m^-), \tag{5.24}$$

the functions ϕ_j satisfy $(\hat{H}_m^- - E_{m,j})\phi_j = \psi_j$, $\big(\hat{H}_m^- - E_{m,j}\big)\psi_j = 0$. Hence, the Hamiltonian $\hat{H}_m^-$ restricted on the kernel of $\hat{Y}_{m,l}^-$ contains Jordan blocks associated with the energies E_l. Due to the similar structure of the spectral polynomial of $\hat{Z}_{m,l}^-$ (5.16), the non-physical states annihilated by the supercharge of the hidden bosonized supersymmetry are of the same nature.

It is instructive to discuss the case of $l = 0$ in detail. In this case $\hat{H}_0^+$ corresponds to the free particle. The commutation relations (5.9) tells that the free particle energy operator is related to $\hat{H}_m^-$ via the following intertwining relations:

$$\hat{X}_{m,0}^+\hat{H}_0^+ = \hat{H}_m^-\hat{X}_{m,0}^+, \qquad \hat{Y}_{m,0}^+\hat{H}_0^+ = \hat{H}_m^-\hat{Y}_{m,0}^+, \tag{5.25}$$

$$\hat{X}_{m,0}^-\hat{H}_m^- = \hat{H}_0^+\hat{X}_{m,0}^-, \qquad \hat{Y}_{m,0}^-\hat{H}_m^- = \hat{H}_0^+\hat{Y}_{m,0}^-. \tag{5.26}$$

The first relation of (5.26) is mediated by $\hat{X}_{m,0}^- = \mathcal{D}_1\cdots\mathcal{D}_m$. Keeping in mind (5.22), the supercharge $\hat{X}_{m,0}^-$ annihilates all the bound states of $\hat{H}_m^-$. In the second relation of (5.26), the Hamiltonians are intertwined by $\hat{Y}_{m,0}^- = \mathcal{D}_0\mathcal{D}_1\cdots\mathcal{D}_m$. Apparently, this operator makes the same job as $\hat{X}_{m,0}^-$ but $\hat{Y}_{m,0}^-$ annihilates additionally the first scattering state of $\hat{H}_m^-$.

The operator $\hat{X}_{m,0}^-$ transforms the lowest scattering state of $\hat{H}_m^-$ into a constant function, the scattering state of a free particle corresponding to the lowest energy. This function is annihilated by $\hat{Y}_{m,0}^+$ while applying $\hat{X}_{m,0}^+$ we get the initial first scattering state of $\hat{H}_m^-$. In general, the operators $\hat{X}_{m,0}^+$ and $\hat{Y}_{m,0}^+$ transform solutions of Schrödinger equation corresponding to the free particle H_0^+ into the (formal) eigenstates of $\hat{H}_m^-$. They can be employed in reconstruction of the scattering states of the Hamiltonian $\hat{H}_m^-$ from the plane wave states of a free particle,

$$\begin{aligned}\psi_\kappa^\pm &= \hat{X}_{m,0}^+\,\mathrm{e}^{\pm\mathrm{i}\kappa x} = \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_{-2}\mathcal{D}_{-1}\,\mathrm{e}^{\pm\mathrm{i}\kappa x}\\ &\propto \hat{Y}_{m,0}^+\,\mathrm{e}^{\pm\mathrm{i}\kappa x} = \mathcal{D}_{-m}\mathcal{D}_{-m+1}\cdots\mathcal{D}_{-1}\mathcal{D}_0\,\mathrm{e}^{\pm\mathrm{i}\kappa x},\end{aligned} \tag{5.27}$$

where $\tilde{\psi}_\kappa^\pm = \mathrm{e}^{\pm\mathrm{i}\kappa x}$ satisfies

$$\hat{H}_0^+\tilde{\psi}_\kappa^\pm = E_\kappa\tilde{\psi}_\kappa^\pm, \qquad \hat{H}_m^-\psi_\kappa^\pm = E_\kappa\psi_\kappa^\pm, \qquad E_k = \kappa^2. \tag{5.28}$$

Let us summarize the obtained results. Extending the real period of the self-isospectral extension of associated Lamé Hamiltonian to infinity, the associated superalgebraic structure was modified. The squares of the supercharges $\mathcal{Z}_{m,l}^{PT}$ and $\mathcal{Y}_{m,l}^{PT}$ turned out to be degenerated polynomials in $\mathcal{H}_{m,l}^{PT}$. This modifies the structure of the underlying superalgebra of the bosonic operators $\mathcal{G}_a^{(\pm)}$ in the dependence on a chosen grading operator Γ_*. Comparing with the periodic case, the bosonic operators form the algebra $u(1)\oplus e(2)\oplus e(2)$ for the singlet energy levels. Recall that the periodic *extended* tri-supersymmetric system does not have singlet states in its spectrum.

The operators $\hat{X}_{m,l}^-$ and $\hat{Y}_{m,l}^-$ annihilate the $m-l$ lowest bound states of the Hamiltonian $\hat{H}_m^-$. The eigenvectors of $\hat{H}_l^+$ corresponding to these energies cease to be physically acceptable. Consequently, isospectrality of the initial system is broken. Speaking in terms of the extended system, the supercharge $\mathcal{X}_{m,l}^{PT}$ annihilates the singlet states. The operator $\mathcal{Y}_m^{PT}$ annihilates both doublets and singlets which are annihilated by the diagonal operator $\mathcal{Z}_m^{PT}$ as well. From this point of view, the spontaneously (or, dynamically) partially broken tri-supersymmetry of the periodic system is recovered in the infinite-period limit.

6. Concluding remarks and outlook

In the particular case of associated Lamé systems, the results of the present paper should be understood in a broader context of the existing literature. Dunne and Feinberg considered the

class $l = m - 1$ of associated Lamé Hamiltonians (4.2) as an example of the self-isospectral extension provided by Darboux transformation [25]. Khare and Shukhatme [27] found that this transformation provides a self-isospectral extension of pure Lamé systems just in the one-gap case while for the other setting the extension proved to be of a completely different nature. On the other hand, Fernández *et al* revealed self-isospectrality of two-gap Lamé Hamiltonian when the second-order transformation was applied [30].

In the light of the presented results, we can understand those findings just as pieces of the mosaic, which was fully unfolded by the structure of the tri-supersymmetry and especially by the self-isospectral supersymmetry of the associated Lamé system. In particular, the system considered by Dunne and Feinberg is the self-isospectral extension $\mathcal{H}_{m,m-1}$ of the associated Lamé Hamiltonian, see (4.21). Besides the first-order supercharge $\mathcal{X}_{m,m-1}$, the list of its local integrals of motions should be completed by the other non-diagonal supercharge $\mathcal{Y}_{m,m-1}$ and diagonal integral $\mathcal{Z}_{m,m-1}$ which plays the role of the central charge of the resulting extended $N = 4$ nonlinear supersymmetry. Although both $\mathcal{X}_{m,m-1}$ and $\mathcal{Z}_{m,m-1}$ annihilate the doublet of ground states, the tri-supersymmetry is spontaneously partially broken since the doublet of ground states does not vanish under the action of the supercharge $\mathcal{Y}_{m,m-1}$. This suggests that the supersymmetry breaking should be analyzed having in mind the complete set of nontrivial local integrals, which are $\mathcal{Z}$, $\mathcal{Q}_{+}^{(a)}$ and $\mathcal{Q}_{-}^{(a)}$ in the case of the studied general class of finite-gap systems.

On our way to the presented results we left untouched various appealing questions and problems. For instance, the self-isospectrality conjecture could be tested on the finite-gap systems with missing anti-periodic states. Since these should be prevented from the self-isospectral extensions, the structure of the tri-supersymmetry could exhibit peculiarities in this case. Besides, the exact proof of the conjecture should be provided.

The infinite-period limit could be an effective technique in production of the tri-supersymmetric systems with non-periodic potentials. In the limit case of the self-isospectral extension of the associated Lamé Hamiltonian, the isospectrality was broken followed by the recovery of the exact tri-supersymmetry. There appears a natural question whether this is the common feature or there exist isospectral extensions of non-periodic systems with broken tri-supersymmetry. The limit of other isospectral extensions of associated Lamé system could provide an insight into the general situation. The relation of the tri-supersymmetry and the representations of Lie algebras might give an interesting insight into involved physical models as well.

Our construction of the tri-supersymmetric extensions was based on the specific factorization of the odd-order integral of motion. Relaxing the smoothness of the potential, the formal construction should be applicable on the broad family of algebro-geometric potentials where the presence of the parity-odd diagonal integral $\mathcal{Z}$ is guaranteed. It is worth mentioning the Treibich–Verdier family of potentials [68] in this context. Besides the associated Lamé systems, this family contains singular potentials, which could be convenient examples to study the tri-supersymmetry in singular systems.

Regular Crum–Darboux transformations with zero modes in the prohibited bands can produce self-isospectral potentials with a generic shift of the coordinate, or superpartners with periodicity defects. The particular results of this type were obtained in [28, 29, 44–46] by making use of the first- and the second-order transformations applied to one- and two-gap Lamé equations. It would be interesting to analyze such a class of systems on the presence of the tri-supersymmetric structure.

The revealed supersymmetric structure was based on the internal properties of the integral of motion Z, related to the KdV hierarchy. This indication of the tri-supersymmetry and self-isospectrality in the context of nonlinear integrable systems should be followed and analyzed.

A special attention should be paid to possible manifestations of the tri-supersymmetry in physical systems [34].

Acknowledgments

The work has been partially supported by CONICYT, DICYT (USACH) and by FONDECYT under grants 1050001 and 3085013. We are grateful to V Enolskii, V Spiridonov, A Treibich, R Weikard and A Zabrodin for valuable communications. Our special thanks are to B Dubrovin for many detailed explanations on the theory of finite-gap systems.

Appendix A

Higher-order differential operators play the key role in the construction of the tri-supersymmetry since they mediate intertwining of the superpartner Hamiltonians. As we explained in the section on the Crum–Darboux transformations, the properties of these operators are determining for the physical characteristics of the superpartner systems. We present here a short resume of the relevant facts referring for the details to [38, 40].

Consider a differential operator of order n which annihilates n functions ψ_i, $i = 1, \dots, n$,

$$A_n = D^n + \sum_{j=1}^{n} c_j^A(x) D^{n-j}, \qquad A_n \psi_i = 0, \qquad i = 1, \dots, n. \tag{A.1}$$

Its coefficients are determined by the functions ψ_i. For instance, the coefficient $c_1^A(x)$ can be given in terms of the Wronskian of the n functions ψ_i, $c_1^A(x) = -\frac{\mathrm{d}}{\mathrm{d}x} \ln W(\psi_1, \dots, \psi_n)$, where $W(\psi_1, \dots, \psi_n) = W = \det B$, $B_{i,j} = \frac{\mathrm{d}^{j-1}\psi_i}{\mathrm{d}x^{j-1}}$, $i, j, = 1, \dots, n$. This is in accordance with the general form for the coefficients $c_j^A(x) = -\frac{W_j}{W}$, $j = 1, \dots, n$, where W_j is the determinant of the matrix B modified by replacing the line $\psi_1^{(n-j)}, \dots, \psi_n^{(n-j)}$ by $\psi_1^{(n)}, \dots, \psi_n^{(n)}$. In this notation, $W_0 \equiv W$.

The operator A_n can be factorized in terms of the first-order differential operators. There follow equivalent representations of A_n which provide a better insight into the properties of the operator, see [38],

$$A_n = (-1)^n \frac{W_n}{W_{n-1}} D \frac{W_{n-1}^2}{W_n W_{n-2}} D \cdots D \frac{W_1^2}{W_2 W_0} D \frac{W_0}{W_1}. \tag{A.2}$$

We can write equivalently

$$A_n = L_n L_{n-1} \cdots L_2 L_1, \qquad L_j = D - \alpha_j, \qquad \alpha_j = \frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{W_j}{W_{j-1}}, \qquad j = 1, \dots, n. \tag{A.3}$$

The operator can also be expressed as a determinant

$$A_n = W^{-1}(\psi_1, \dots, \psi_n) \begin{vmatrix} \psi_1 & \psi_2 & \cdots & \psi_n & 1 \\ \psi_1' & \psi_2' & \cdots & \psi_n' & D \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \psi_1^{(n-1)} & \psi_2^{(n-1)} & \cdots & \psi_n^{(n-1)} & D^{n-1} \\ \psi_1^{(n)} & \psi_2^{(n)} & \cdots & \psi_n^{(n)} & D^n \end{vmatrix}, \tag{A.4}$$

where the multiplicative factor fixes the coefficient of D^n to be equal to 1. Here, the determinant of the operator-valued $(n + 1) \times (n + 1)$ matrix is defined as $\det C =$

Table 1. Integrals of motion and structure polynomials, grading $\Gamma_* = \sigma_3$.

Fermionic integrals	$F_1 = \mathcal{Q}_-$	$F_2 = -R\sigma_3\mathcal{Q}_-$	$F_3 = -\mathrm{i}R\mathcal{Q}_-$	$F_4 = \mathrm{i}\sigma_3\mathcal{Q}_-$
	$F_5 = R\mathcal{Q}_+$	$F_6 = -\mathcal{Q}_+$	$F_7 = \mathrm{i}\sigma_3\mathcal{Q}_+$	$F_8 = -\mathrm{i}R\sigma_3\mathcal{Q}_+$
Bosonic integrals	$\mathcal{H}$	$\Sigma_1 = -R$	$\Gamma_* = \sigma_3$	$\Sigma_2 = -R\sigma_3$
	$B_1 = -\mathrm{i}R\sigma_3\mathcal{Z}$	$B_2 = -\sigma_3\mathcal{Z}$	$B_3 = -\mathrm{i}R\mathcal{Z}$	$B_4 = -\mathcal{Z}$
Polynomials	$P_{22} = P_{2B} = P_-(\mathcal{H})$	$P_{11} = P_{1B} = P_+(\mathcal{H})$	$P_{12} = 1$	$P_B = P_Z(\mathcal{H})$

$\sum_{\sigma\in G_{n+1}} \mathrm{sgn}(\sigma) C_{\sigma(1),1} C_{\sigma(2),2} \cdots C_{\sigma(n+1),n+1}$, where G_{n+1} is a set of all possible permutations of the integers $\{1, \ldots, n+1\}$.

Particularly, when ψ_i are periodic functions except even number of antiperiodic ones, the Wronskian W is periodic. Since the derivatives do not change the period of the functions, W_i are periodic as well. The formulae above then justify periodicity of the operator A_n.

Finally, let us make a few comments on the superpartner Hamiltonians H and $\tilde{H}$ intertwined by operator A_n (see (3.10)) which annihilates a part of the physical states of H. Let the potential of H be smooth and the Wronskian W computed on the kernel of A_n be a nodeless function. Then the potential of $\tilde{H}$ is smooth as well. The operator A_n can be used in reconstruction of the eigenstates $\tilde{\psi}$ of $\tilde{H}$ corresponding to the eingenstates $\psi \neq \psi_i$ of H with the same eigenvalue, $\tilde{H}\tilde{\psi} = E\tilde{\psi}$, $H\psi = E\psi$, $\tilde{\psi} = A_n\psi$. These wavefunctions $\tilde{\psi}$ can be also represented as

$$\tilde{\psi} = A_n\psi = \frac{W(\psi_1, \ldots, \psi_n, \psi)}{W(\psi_1, \ldots, \psi_n)}. \tag{A.5}$$

This receipt fails in the reconstruction of the states $\tilde{\psi}_i$, which correspond to the same eigenvalue as ψ_i, $\tilde{H}\tilde{\psi}_i = E_i\tilde{\psi}_i$, $H\psi_i = E_i\psi_i$, where ψ_i is annihilated by A_n. These functions $\tilde{\psi}_i$, annihilated by $A_n^\dagger$, are given by

$$\tilde{\psi}_i = \frac{W(\psi_1, \ldots, \hat{\psi}_i, \ldots, \psi_n,)}{W(\psi_1, \ldots, \psi_n)}, \qquad i = 1, \ldots, n, \qquad A_n^\dagger\tilde{\psi}_i = 0, \tag{A.6}$$

where the entry below a symbol '^' is omitted.

Appendix B

Grading $\Gamma_* = \sigma_3$

In this case, which corresponds to the usual choice of the grading operator, the non-diagonal supercharges $\mathcal{Q}_\pm$ are fermionic operators, $\{\mathcal{Q}_\pm, \sigma_3\} = 0$, whereas the diagonal integral $\mathcal{Z}$ is a bosonic generator. Table 1 represents the explicit identification of the bosonic and fermionic generators, and the corresponding polynomials appearing in the (anti)commutation relations.

Grading $\Gamma_* = R$

For the choice $\Gamma_* = R$, the parity-odd diagonal, $\mathcal{Z}$, and non-diagonal, $\mathcal{Q}_-$, integrals are fermionic supercharges. The non-diagonal parity-even integral $\mathcal{Q}_+$ is identified as a bosonic generator. The identification of all the generators and structure polynomials are given by table 2.

Table 2. Integrals of motion and structure polynomials, grading $\Gamma_* = R$.

Fermionic integrals	$F_1 = \mathcal{Q}_-$	$F_2 = R\sigma_3\mathcal{Q}_-$	$F_3 = \mathrm{i}\sigma_3\mathcal{Q}_-$	$F_4 = \mathrm{i}R\mathcal{Q}_-$
	$F_5 = \mathrm{i}R\mathcal{Z}$	$F_6 = \mathrm{i}R\sigma_3\mathcal{Z}$	$F_7 = \sigma_3\mathcal{Z}$	$F_8 = \mathcal{Z}$
Bosonic integrals	$\mathcal{H}$	$\Gamma_* = R$	$\Sigma_1 = \sigma_3$	$\Sigma_2 = R\sigma_3$
	$B_1 = \mathcal{Q}_+$	$B_2 = \mathrm{i}\sigma_3\mathcal{Q}_+$	$B_3 = R\mathcal{Q}_+$	$B_4 = \mathrm{i}R\sigma_3\mathcal{Q}_+$
Polynomials	$P_{22} = P_{12} = P_-(\mathcal{H})$	$P_{11} = P_Z(\mathcal{H})$	$P_{2B} = 1$	$P_B = P_{1B} = P_+(\mathcal{H})$

Table 3. Integrals of motion and structure polynomials, grading $\Gamma_* = R\sigma_3$.

Fermionic integrals	$F_1 = \mathcal{Z}$	$F_2 = -\sigma_3\mathcal{Z}$	$F_3 = -\mathrm{i}R\mathcal{Z}$	$F_4 = \mathrm{i}R\sigma_3\mathcal{Z}$
	$F_5 = R\mathcal{Q}_+$	$F_6 = -\mathcal{Q}_+$	$F_7 = \mathrm{i}R\sigma_3\mathcal{Q}_+$	$F_8 = -\mathrm{i}\sigma_3\mathcal{Q}_+$
Bosonic integrals	$\mathcal{H}$	$\Sigma_1 = -R$	$\Sigma_2 = -\sigma_3$	$\Gamma_* = R\sigma_3$
	$B_1 = -\mathrm{i}\sigma_3\mathcal{Q}_-$	$B_2 = -R\sigma_3\mathcal{Q}_-$	$B_3 = -\mathrm{i}R\mathcal{Q}_-$	$B_4 = -\mathcal{Q}_-$
Polynomials	$P_{22} = P_Z(\mathcal{H})$	$P_{11} = P_{12} = P_+(\mathcal{H})$	$P_{1B} = 1$	$P_B = P_{2B} = P_-(\mathcal{H})$

Table 4. Fermion–fermion anti-commutation relations. Here the overall multiplicative factor 2 is omitted. To get anti-commutator, the corresponding entry should be multiplied by 2, for instance, $\{F_1, F_1\} = 2P_{22}$.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
F_1	P_{22}	$\Sigma_2 P_{22}$	0	0	0	$B_4 P_{12}$	0	$B_1 P_{12}$
F_2	$\Sigma_2 P_{22}$	P_{22}	0	0	$-B_2 P_{12}$	0	$-B_3 P_{12}$	0
F_3	0	0	P_{22}	$\Sigma_2 P_2$	0	$-B_3 P_{12}$	0	$B_2 P_{12}$
F_4	0	0	$\Sigma_2 P_{22}$	P_{22}	$B_1 P_{12}$	0	$-B_4 P_{12}$	0
F_5	0	$-B_2 P_{12}$	0	$B_1 P_{12}$	P_{11}	$\Sigma_1 P_{11}$	0	0
F_6	$B_4 P_{12}$	0	$-B_3 P_{12}$	0	$\Sigma_1 P_{11}$	P_{11}	0	0
F_7	0	$-B_3 P_{12}$	0	$-B_4 P_{12}$	0	0	P_{11}	$\Sigma_1 P_{11}$
F_8	$B_1 P_{12}$	0	$B_2 P_{12}$	0	0	0	$\Sigma_1 P_{11}$	P_{11}

Table 5. Boson–fermion commutation relations. The overall multiplicative factor 2 is omitted. To get commutator, the corresponding entry should be multiplied by 2, for instance, $[\Gamma_*, F_1] = -2\mathrm{i}F_4$.

	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
Γ_*	$-\mathrm{i}F_4$	$-\mathrm{i}F_3$	$\mathrm{i}F_2$	$\mathrm{i}F_1$	$\mathrm{i}F_8$	$\mathrm{i}F_7$	$-\mathrm{i}F_6$	$-\mathrm{i}F_5$
Σ_1	$-\mathrm{i}F_3$	$-\mathrm{i}F_4$	$\mathrm{i}F_1$	$\mathrm{i}F_2$	0	0	0	0
Σ_2	0	0	0	0	$\mathrm{i}F_7$	$\mathrm{i}F_8$	$-\mathrm{i}F_5$	$-\mathrm{i}F_6$
B_1	0	$\mathrm{i}F_6 P_{2B}$	$-\mathrm{i}F_7 P_{2B}$	0	0	$-\mathrm{i}F_2 P_{1B}$	$\mathrm{i}F_3 P_{1B}$	0
B_2	$\mathrm{i}F_7 P_{2B}$	0	0	$\mathrm{i}F_6 P_{2B}$	0	$-\mathrm{i}F_4 P_{1B}$	$-\mathrm{i}F_1 P_{1B}$	0
B_3	$-\mathrm{i}F_5 P_{2B}$	0	0	$\mathrm{i}F_8 P_{2B}$	$\mathrm{i}F_1 P_{1B}$	0	0	$-\mathrm{i}F_4 P_{1B}$
B_4	0	$-\mathrm{i}F_8 P_{2B}$	$-\mathrm{i}F_5 P_{2B}$	0	$\mathrm{i}F_3 P_{1B}$	0	0	$\mathrm{i}F_2 P_{1B}$

Grading $\Gamma_* = R\sigma_3$

With this choice of the grading operator, integrals $\mathcal{Z}$ and $\mathcal{Q}_+$ are identified as fermionic supercharges, integral $\mathcal{Q}_-$ is a bosonic generator. Complete identification of the generators and structure polynomials are represented by table 3.

The anti-commutation relations between the fermionic operators are given in table 4, while table 5 provides the boson–fermion commutation relations.

Appendix C

In the treatment of the section 4.1, we left untouched the system described by Lamé associated Hamiltonian with $m = l$. Due to an identity $\mathrm{dn}(x + K) = k'/\mathrm{dn}\, x$, in contrary to the other members of the family, its period is K. This fact explains why the algebraic methods applied to other members of the family in this case say that the dimension of $sl(2, \mathbb{R})$ representation realized on antiperiodic in the period $2K$ singlet states is equal to $m - l = 0$. This is just because singlet states with such a period does not exist. The place of this system in the mosaic of the tri-supersymmetric self-isospectral systems is clarified by its intimate relation to pure Lamé system, mediated by Landen transformation [69, 70].

Landen's transformation of the elliptic functions can be written as

$$\begin{aligned} \mathrm{sn}(x,k) &= \alpha \frac{\mathrm{sn}\left(\frac{x}{\alpha},\kappa\right)\mathrm{cn}\left(\frac{x}{\alpha},\kappa\right)}{\mathrm{dn}\left(\frac{x}{\alpha},\kappa\right)}, \qquad \mathrm{cn}(x,k) = \frac{1-\alpha\mathrm{sn}^2\left(\frac{x}{\alpha},\kappa\right)}{\mathrm{dn}\left(\frac{x}{\alpha},\kappa\right)}, \\ \mathrm{dn}(x,k) &= \frac{\kappa' + (2-\alpha)\,\mathrm{cn}^2\left(\frac{x}{\alpha},\kappa\right)}{\mathrm{dn}\left(\frac{x}{\alpha},\kappa\right)}, \end{aligned} \tag{C.1}$$

where $\alpha = \frac{2}{1+k}$, $\kappa^2 = \frac{4k}{(1+k)^2}$, $k = \frac{1-\kappa'}{1+\kappa'}$, $\alpha = 1+\kappa'$. To avoid confusions, let us denote explicitly the dependence of the complete elliptic integral K on the modular parameter such that we will write $K(k)$ or $K(\kappa)$. Since $K(\kappa) = (1+k)K(k)$, Landen's transformation divides in two the period $2K(k)$ of the elliptic functions in the sense that the period of the resulting expression is $K(\kappa)$.

Using the identities (C.1), we can rewrite the Lamé Hamiltonian in terms of the elliptic functions of a new variable $y = \frac{x}{\alpha}$ and the modular parameter κ, $H^-_{m,0}(x,k) = \frac{1}{\alpha^2}[H_{m,m}(y,\kappa)] + \text{const}$. The displacement $K(k)$ of the pure Lamé tri-supersymmetric partner changes to $\frac{K(\kappa)}{2}$ in the case $m = l$. It is in accordance with our result on the general case $m \neq l$ where the superpartner potential was displaced in the half of the real period. Thus, we obtain finally the relation

$$\begin{pmatrix} H^-_{m,0}(x+K(k),k) & 0 \\ 0 & H^-_{m,0}(x,k) \end{pmatrix} = \frac{1}{\alpha^2}\begin{pmatrix} H^-_{m,m}\left(y+\frac{K(\kappa)}{2},\kappa\right) & 0 \\ 0 & H^-_{m,m}(y,\kappa) \end{pmatrix} + c, \tag{C.2}$$

where c is a constant term.

It suggests directly the form of the tri-supersymmetry in the special case of $m = l$ Lamé associated systems; all the operators commuting with $\mathcal{H}_{m,0}$ commute with $\mathcal{H}_{m,m}$ as well. To get their explicit form for the systems described by $\mathcal{H}_{m,m}$ we just have to rescale the variable and apply the identities (C.1) in formulae (4.21) for $\mathcal{X}_{m,0}(x,k)$, $\mathcal{Y}_{m,0}(x,k)$ and $\mathcal{Z}_{m,0}(x,k)$. Then we can write immediately

$$\begin{aligned} \mathcal{X}_{m,m}(y,\kappa) &= \mathcal{X}_{m,0}(\alpha y, k(\kappa)) = \mathcal{X}_{m,0}\left((1+\kappa')y, \frac{1-\kappa'}{1+\kappa'}\right), \\ \mathcal{Y}_{m,m}(y,\kappa) &= \mathcal{Y}_{m,0}(\alpha y, k(\kappa)) = \mathcal{Y}_{m,0}\left((1+\kappa')y, \frac{1-\kappa'}{1+\kappa'}\right), \\ \mathcal{Z}_{m,m}(y,\kappa) &= \mathcal{Z}_{m,0}(\alpha y, k(\kappa)) = \mathcal{Z}_{m,0}\left((1+\kappa')y, \frac{1-\kappa'}{1+\kappa'}\right). \end{aligned} \tag{C.3}$$

Naturally, the algebraic relations between the operators remain unchanged. Thus, the self-isospectral supersymmetry and associated superalgebra are recovered for $m = l$ case.

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